

A Simplified Embedding of \mathbf{E} into Monomodal \mathbf{K}^*

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Abstract

In this paper we will provide a modal-to-modal translational embedding of \mathbf{E} into \mathbf{K} , simplifying a similar result which is obtainable using a novel translation due to S.K. Thomason.

1 Introduction

Throughout this paper we will be concerned with translational embeddings which faithfully embed the smallest congruential modal logic \mathbf{E} into the smallest normal modal logic \mathbf{K} . In [6] it is shown that we can faithfully embed \mathbf{E} into trimodal \mathbf{K} using the modal-to-modal translation $(\cdot)^F$ for which $(\Box A)^F = \Diamond_1(\Box_2(A)^F \wedge \Box_3\neg(A)^F)$.¹ Here we are to think of \Diamond_1 as quantifying over neighborhoods, \Box_2 as quantifying within neighborhoods, and \Box_3 as quantifying within their complements – thus allowing us to mimic the truth conditions for \Box -formulas within a neighborhood model. In [3] it is noted that we can simplify this translation to one which faithfully embeds \mathbf{E} in bimodal \mathbf{K} by making \Box_1 and \Box_2 the same operator – thus making the new translation be such that $(\Box A)^{F'} = \Diamond_1(\Box_1(A)^{F'} \wedge \Box_2\neg(A)^{F'})$. One might wonder then whether a further simplification along these lines – identifying \Box_1 and \Box_2 in the clause for $(\Box A)^{F'}$ – would allow us to faithfully embed \mathbf{E} in monomodal \mathbf{K} . As it happens the simplification cannot be of this nature – as the translation $(\cdot)^{F''}$ for which $(\Box A)^{F''} = \Diamond(\Box(A)^{F''} \wedge \Box\neg(A)^{F''})$ fails to faithfully embed \mathbf{E} into \mathbf{K} , there being \mathbf{K} -provable formulas of the form $(A)^{F''}$ for which A is not \mathbf{E} -provable.² Consequently we will have to look elsewhere for a translation which faithfully embeds \mathbf{E} into monomodal \mathbf{K} .

It turns out that one way for us to find such a result is to use a very general result of Thomason which allows us to faithfully embed bimodal logics into a class of normal monomodal logics. This particular result when taken together

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¹Here n -modal \mathbf{K} (alias \mathbf{K}_n) is the modal logic built on the propositional language containing n box operators \Box_1, \dots, \Box_n each of which obeys all the inferential principles of \Box in monomodal \mathbf{K} .

²For example, let A be the formula “ $\Box p \leftrightarrow \Box\neg p$ ”, whose $(\cdot)^{F''}$ -translation is provable in every congruential modal logic.

with the embedding of \mathbf{E} into bimodal \mathbf{K} due to [3], allows us to produce a modal-to-modal translation which faithfully embeds \mathbf{E} into \mathbf{K} . The modal complexity of the resulting translation is quite high, and the result somewhat indirect. Thus, after having given the Thomason derived translation in §2, we will go on (in §3) to produce our own simplified translation of \mathbf{E} into \mathbf{K} inspired by some work by [1]. Therein Brown claimed that the translation which replaces all occurrences of \Box with $\Diamond\Box$ (what we call $\tau_{\Diamond\Box}$ below) faithfully embeds \mathbf{E} into monomodal \mathbf{K} . As it happens though, this translation faithfully embeds \mathbf{EM} into \mathbf{K} – where \mathbf{EM} is the smallest congruential extension of \mathbf{E} by the formula ‘ $\Box(A \wedge B) \rightarrow \Box A$ ’.³

Before we continue an explanation is due concerning some terminology. Let \mathbf{S} and \mathbf{S}' be modal logics in the sets of formulas sense,⁴ and let a translation τ be a function from the language of \mathbf{S} to the language of \mathbf{S}' which *faithfully embeds \mathbf{S} into \mathbf{S}'* whenever the following holds for all formulas A :

$$A \in \mathbf{S} \text{ if and only if } \tau(A) \in \mathbf{S}'.$$

Let us say that a translation τ is *variable-fixed* if $\tau(p_i) = p_i$, and *homonymous on the classical connectives* whenever $\tau(\#(A_1, \dots, A_n)) = \#(\tau(A_1), \dots, \tau(A_n))$ for all classical connectives $\#$. In particular we will be interested in translations which fulfill both of these conditions and translate $\Box A$ in terms of some formula containing exactly one variable (p) $C(p)$ such that $\tau(\Box A) = C(\tau(A))$. These translations are easily seen to be definitional in the sense of [9]. For more information on such modal-to-modal translations the reader is referred to [10], from whom we take the convention of associating with each such formula $C(p)$ a translation τ_C which replaces all occurrences of $\Box A$ in a formula with $C(\tau_C(A))$.

Given a formula in a single propositional variable $C(p)$ – a unary context – and a modal logic \mathbf{S} , we will say that $C(p)$ is *congruential according to \mathbf{S}* if $C(A) \leftrightarrow C(B) \in \mathbf{S}$ whenever $A \leftrightarrow B \in \mathbf{S}$. A modal logic \mathbf{S} is *congruential* whenever the context $\Box p$ is congruential according to \mathbf{S} . Following [2] we will denote the smallest congruential modal logic by \mathbf{E} . It is well known that \mathbf{E} is determined by the class of all neighborhood frames, where a *neighborhood frame* is a structure $\langle W, N \rangle$ where $W \neq \emptyset$ and N is a function from W to $\wp(\wp(W))$. The definition of truth at a point in a neighborhood model $\langle W, N, V \rangle$ is standard for the non-modal connectives, with the clause for $\Box A$ being as follows, for a formula A and a point $x \in W$.

$$\langle W, N, V \rangle \models_x \Box A \text{ if and only if } \llbracket A \rrbracket \in N(x).$$

Here $\llbracket A \rrbracket$ is the set of all points $y \in W$ at which the formula A is true. It is worth also recalling the notion of the modal degree of a formula A – the

³Brown does not claim this as directly as we have above. On p. 14 he claims that the logic of one of the operators his bimodal logic of action and ability is \mathbf{E} , and on pp.15-16 he shows that the translation $\tau_{\Diamond\Box}$ will faithfully embed this subsystem into \mathbf{K} . It is the first claim which is incorrect.

⁴That is to say, sets of formulas of the propositional language based on some set of functionally complete classical connectives and the unary connective \Box , closed under Modus Ponens and Uniform Substitution.

maximum degree of nesting of modal operators. Define the *modal degree of a formula* A inductively as follows: $d(p_i) = 0$; $d(\neg A) = d(A)$; $d(A \rightarrow B) = d(A \wedge B) = d(A \vee B) = \max(d(A), d(B))$; $d(\Box A) = d(A) + 1$.⁵

2 Thomason's Translation

In [6], building on some work done in the 1970s by S.K. Thomason ([7], [8]), we are presented with a general account of how to faithfully embed all bimodal normal modal logics into a class of monomodal normal modal logics. In this section we will begin by presenting these results, and then show how we can use them to give a faithful embedding of \mathbf{E} into monomodal \mathbf{K} . Throughout we will use $\mathbf{t} = \Box\perp$, $\mathbf{w} = \Diamond\Box\perp$ and $\mathbf{b} = \neg\mathbf{t} \wedge \neg\mathbf{w}$.⁶ The *Thomason translation* – τ_{Tho} – is the following function which maps formulas of the language of bimodal logic to the monomodal language.

$$\begin{aligned} \tau_{\text{Tho}}(p_i) &= p_i \\ \tau_{\text{Tho}}(A \wedge B) &= \tau_{\text{Tho}}(A) \wedge \tau_{\text{Tho}}(B) \\ \tau_{\text{Tho}}(\neg A) &= \neg\tau_{\text{Tho}}(A) \\ \tau_{\text{Tho}}(\Box_1 A) &= \Box(\mathbf{w} \rightarrow \tau_{\text{Tho}}(A)) \\ \tau_{\text{Tho}}(\Box_2 A) &= \Box(\mathbf{b} \rightarrow \Box(\mathbf{b} \rightarrow \Box(\mathbf{w} \rightarrow \tau_{\text{Tho}}(A)))) \end{aligned}$$

We will occasionally use $\Box_w A$ as an abbreviation for $\Box(\mathbf{w} \rightarrow A)$ and $\Box_b A$ as an abbreviation for $\Box(\mathbf{b} \rightarrow A)$ and $\Box_t A$ as an abbreviation for $\Box(\mathbf{t} \rightarrow A)$. It bears noting that this translation is very different from the one given in [7, p.550] – which is neither variable fixed nor homonymous on the propositional connectives, as well as from that given in [4, p.308] – which is variable fixed, but not homonymous on the propositional connectives, translating ‘ $\neg p$ ’ as ‘ $\mathbf{w} \wedge \neg p$ ’.⁷ In what follows we will refer to the translation τ_{Tho} as the ‘Thomason translation’ to avoid possible ambiguity, as this is all we need to derive our first embedding of \mathbf{E} into monomodal \mathbf{K} .

Consider the following formulas (from [6, p.116]).

⁵If we are considering formulas in the language of bimodal logic we can replace the last clause with $d(\Box_i A) = d(A) + 1$.

⁶These labels are taken from [6], where they are mnemonic for ‘terminal’, ‘white’ and ‘black’ respectively – Kracht and Wolter using \Box and \blacksquare for what we’re calling \Box_1 and \Box_2 .

⁷This translation does not do the work which Kracht sets for it – the crucial result (Proposition 6.6.14 of [4]) being incorrect. Consider the model $\mathcal{M} = \langle \{x, y\}, \{\langle x, y \rangle\}, V \rangle$ where $V(p) = \{x\}$. Let $V'(p) = \{x^\circ, x^\bullet\}$ – then V' is a valuation such that $V'(p) \cap W^\circ = (V(p))^\circ$. It is easy to see that $\mathcal{M}^s \models_{x^\circ} \mathbf{w} \wedge \Diamond p$ while $\mathcal{M} \not\models_x \Diamond p$ – giving us a failure of the ‘only if’ direction of Proposition 6.6.14 in [4]. The incorrectness of this crucial result makes all the results there concerning the Thomason translation incorrect. This problem can be avoided if we alter the translation so that it translates p_i as $\mathbf{w} \wedge p_i$ – making it more closely resemble that in [7]. This (corrected) translation appears in [5].

$$\begin{array}{ll}
\mathbf{w} \rightarrow (\diamond_b p \leftrightarrow \square_b p) & \mathbf{b} \rightarrow (\diamond_w p \leftrightarrow \square_w p) \\
(\mathbf{w} \wedge p) \rightarrow \square_b \square_w p & (\mathbf{b} \wedge p) \rightarrow \square_w \square_b p \\
\diamond_t p \rightarrow \square_t p & (\mathbf{w} \wedge \diamond_t p) \rightarrow \square_w \diamond_t p \\
(\mathbf{w} \wedge \diamond_w \diamond_t p) \rightarrow \diamond_t p & (\mathbf{w} \wedge \diamond_t p) \rightarrow \square_b \square_b \square_w \diamond_t p
\end{array}$$

Let **Sim** be the logic obtained by taking the normal extension of **K** by the above formulas.⁸ What the Thomason translation allows us to do is faithfully embed every normal bimodal logic into a normal extension of **Sim**.

Proposition 1. [6, p.122] *For all sets of bimodal formulas Δ , and all bimodal formulas A we have the following.*

$$A \in \mathbf{K}_2 \oplus \Delta \text{ if and only if } \tau_{\text{Tho}}(A) \in \mathbf{Sim} \oplus \{\mathbf{w} \rightarrow \tau_{\text{Tho}}(B) \mid B \in \Delta\}.$$

In the presence of the above result we can see the general strategy behind the title of [6]. The translation τ_{Tho} faithfully embeds bimodal normal modal logics into normal (monomodal) logics, and a wide range of non-normal modal logics can be faithfully embedded into bimodal normal modal logics – and an even wider range can be faithfully embedded into modal logics extending \mathbf{K}_n for $n > 2$, and so if we lift the Thomason translation to full generality then it seems possible that we may be able to faithfully embed all modal logics into normal monomodal logics. What we will now do is, using the strategy outlined above, present a translation which faithfully embeds **E** into monomodal **K**.

In [6] we are only presented with the translation $(\cdot)^{F'}$ (from §1) which faithfully embeds **E** into \mathbf{K}_3 . So lifting the Thomason translation so that it faithfully embeds trimodal normal modal logics into normal monomodal logics we could obtain a translation which faithfully embeds **E** into **K**. Already at this stage there is an obvious simplification we could make, which does not require us to deal with trimodal logics.

Recall the following result (mentioned above in §1):

Theorem 1. [3, p.307]

$$A \in \mathbf{E} \text{ if and only if } (A)^{F'} \in \mathbf{K}_2.$$

This result, coupled with Proposition 1 allows us to faithfully embed **E** into the logic **Sim** using the translation $\tau_{\text{Tho}}((\cdot)^{F'})$. What we will now show is that this also allows us to show that **E** can be faithfully embedded into **K**.

Theorem 2.

$$A \in \mathbf{E} \text{ if and only if } \tau_{\text{Tho}}((A)^{F'}) \in \mathbf{K}.$$

⁸The label **Sim** is taken from Kracht and Wolter, who use it because they call a translation faithfully embedding **S** into **S'** a simulation of **S** by **S'**.

Proof. For the ‘only if’ direction suppose that $A \notin \mathbf{E}$. Then by Theorem 1 it follows that $(A)^{F'} \notin \mathbf{K}_2$. By Proposition 1 it then follows that $\tau_{\text{Tho}}((A)^{F'}) \notin \mathbf{Sim}$. As $\mathbf{K} \subseteq \mathbf{Sim}$ it follows then that $\tau_{\text{Tho}}((A)^{F'}) \notin \mathbf{K}$.

The ‘if’ direction follows by induction upon the length of derivations of A , the only case of interest coming in the inductive step where A follows from the congruentiality of \Box . But as all contexts are congruential in \mathbf{K} it follows that if $A \leftrightarrow B \in \mathbf{K}$ then $C(A) \leftrightarrow C(B) \in \mathbf{K}$ for the case where $C(p) = \tau_{\text{Tho}}(\Box p)^F$. \square

This gives us a translation τ where $\tau(\Box A) =$

$$\begin{aligned} & \diamond(\diamond\Box\perp \wedge [\Box(\diamond\Box\perp \rightarrow \tau(A)) \wedge \\ & \quad \Box((\neg\diamond\Box\perp \wedge \neg\Box\perp) \rightarrow \Box((\neg\diamond\Box\perp \wedge \neg\Box\perp) \rightarrow \Box(\diamond\Box\perp \rightarrow \neg\tau(A)))]). \end{aligned}$$

As we can see above, the context which we are using here to translate \Box is quite unwieldy, having a modal degree of 6. What we will do in the next section is to provide a simpler, and more direct, faithful embedding of \mathbf{E} into \mathbf{K} .

3 A Simplified Translation of \mathbf{E} into \mathbf{K}

Let $\tau_{\Box'}$ be the modal translation which uniformly replaces all occurrences of $\Box B$ with $\Box'\tau_{\Box'}(B)$, where \Box' is defined as follows.⁹

$$\Box' A =_{Def} \diamond(\diamond(\Box A \wedge \Box\diamond\Box\top) \wedge \diamond(\diamond(\Box\neg A \wedge \Box\diamond\top) \wedge \diamond\diamond\Box\perp)$$

Like every context, the context $C(p) = \Box'p$ is congruential in \mathbf{K} , allowing us to conclude the following.

Lemma 1. *For all formulas A if $A \in \mathbf{E}$ then $\tau_{\Box'}(A) \in \mathbf{K}$.*

Before we prove that $\tau_{\Box'}$ faithfully embeds \mathbf{E} into \mathbf{K} we will first need to recall a result concerning classes of neighborhood frames which determine \mathbf{E} . Say that a neighborhood frame $\langle W, N \rangle$ has *non-empty neighborhoods* whenever for all $x \in W$ we have that $N(x) \neq \emptyset$.

Theorem 3. *\mathbf{E} is determined by the class of all neighborhood frames with non-empty neighborhoods*

Proof. Follows from the fact noted in [2, p.255] that the neighborhood function $N(x) = \{\|A\| : \Box A \in x\} \cup \Delta$ is a canonical neighborhood function whenever $\Delta \subseteq \{X \subseteq W : X \neq \|A\| \text{ for any formula } A\}$. \square

Given a neighborhood model $\mathcal{N} = \langle W, N, V \rangle$, a point $x \in W$ and a neighborhood $X \in N(x)$ let $\langle x, X, i \rangle$ ($0 \leq i \leq 5$) be new points not belonging to W , which we will write as $\langle x, X \rangle^*$, $\langle x, X \rangle^+$, $\langle x, X \rangle^-$, $\langle x, X \rangle^I$, $\langle x, X \rangle_{e1}^I$, $\langle x, X \rangle_{e2}^I$.

⁹The first conjunct of the conjunction in the scope of the main ‘ \diamond ’ here could more simply be written as $\diamond\Box(A \wedge \Box\diamond\top)$, but the formulation above has the advantage of displaying the second conjuncts of the two inner conjunctions as being each other’s negations.

Here we are thinking of the labels $\langle x, X \rangle^+$ and $\langle x, X \rangle^-$ as denoting the neighborhood $\langle x, X \rangle$ and its complement respectively. The superscript I should be read as ‘intermediary’, and the subscripted e in $e1$ and $e2$ as ‘end’. Our reason for the choice of these names should become clear in what follows. Let $K_{\langle x, X \rangle}$ and $R_{\langle x, X \rangle}$ be defined as follows.

$$\begin{aligned} K_{\langle x, X \rangle} &= \{\langle x, X \rangle^*, \langle x, X \rangle^+, \langle x, X \rangle^i, \langle x, X \rangle_{e1}^I, \langle x, X \rangle_{e2}^I, \langle x, X \rangle^-\}. \\ R_{\langle x, X \rangle} &= \{\langle \langle x, X \rangle^*, \langle x, X \rangle^+ \rangle, \langle \langle x, X \rangle^*, \langle x, X \rangle^I \rangle, \langle \langle x, X \rangle^I, \langle x, X \rangle^- \rangle, \\ &\quad \langle \langle x, X \rangle^I, \langle x, X \rangle_{e1}^I \rangle, \langle \langle x, X \rangle_{e1}^I, \langle x, X \rangle_{e2}^I \rangle\}. \end{aligned}$$

Definition 4. Given a neighborhood model with non-empty neighborhoods $\mathcal{N} = \langle W, N, V \rangle$ construct the Kripke model $\mathcal{N}_{EK} = \langle W_{EK}, R_{EK}, V_{EK} \rangle$ as follows.

- $W_{EK} := W \cup \bigcup_{x \in W} \{K_{\langle x, X \rangle} \mid X \in N(x)\}$.
- $R_{EK} := \bigcup_{x \in W} \{R_{\langle x, X \rangle} \mid X \in N(x)\} \cup \{\langle x, \langle x, X \rangle^* \rangle \mid X \in N(x)\} \cup \{\langle \langle x, X \rangle^+, y \rangle \mid y \in X\} \cup \{\langle \langle x, X \rangle^-, y \rangle \mid y \in (W \setminus X)\}$.
- $V_{EK} := V$.

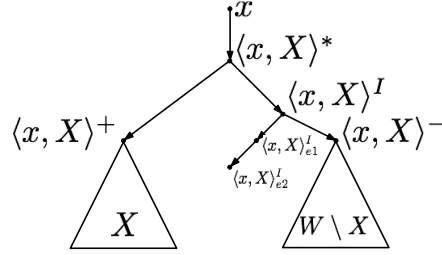


Figure 1: A snapshot of \mathcal{N}_{EK} for a neighborhood $X \in N(x)$.

To get a feel for the workings of this model construction, inspired by those in [1] and [3], see Figure 1 which illustrates what happens to a point $x \in W$ which has X as one of its neighborhoods. The model construction creates a structure like that in Figure 1 for each neighborhood X in $N(x)$ for all points $x \in W$ – the points $\langle x, X \rangle^*$ acting as ‘overseers’ allowing us to check whether some formula B is true throughout some neighborhood X and false throughout its complement $W \setminus X$.

One might wonder about the purpose of the ‘end’ points in the above model construction. What we will now show is that these dead end points, along with the pure formulas in the definition of $\Box' A$, allow us to force the two conjunctions within the scope of the outermost diamond to be true at specific points in the model ($\langle x, X \rangle^+$ and $\langle x, X \rangle^I$ respectively) whenever $\Box' A$ is true at $\langle x, X \rangle^*$.

Lemma 2. For all formulas A and all $x \in W$ and $X \in N(x)$ we have the following.

$$\mathcal{N}_{EK} \models_{\langle x, X \rangle^*} \diamond(\Box A \wedge \Box \Box \diamond \top) \Rightarrow \mathcal{N}_{EK} \models_{\langle x, X \rangle^+} \Box A \wedge \Box \Box \diamond \top.$$

Proof. Suppose that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^*} \diamond(\Box A \wedge \Box \Box \diamond \top)$, and suppose for a reductio that $\mathcal{N}_{EK} \not\models_{\langle x, X \rangle^+} \Box A \wedge \Box \Box \diamond \top$. Then, as the only other point R_{EK} -accessible to $\langle x, X \rangle^*$ is $\langle x, X \rangle^I$ we know that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^I} \Box A \wedge \Box \Box \diamond \top$. In particular this means that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^I} \Box \Box \diamond \top$. Thus $\mathcal{N}_{EK} \models_{\langle x, X \rangle_{e_1}^I} \Box \diamond \top$ and $\mathcal{N}_{EK} \models_{\langle x, X \rangle_{e_2}^I} \Box \diamond \top$, which cannot happen. Thus it follows that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^+} \Box A \wedge \Box \Box \diamond \top$ as desired. \square

Lemma 3. For all formulas A and all $x \in W$ and $X \in N(x)$ we have the following.

$$\begin{aligned} \mathcal{N}_{EK} \models_{\langle x, X \rangle^*} \diamond(\diamond(\Box \neg A \wedge \Box \diamond \top) \wedge \diamond \diamond \Box \perp) \Rightarrow \\ \mathcal{N}_{EK} \models_{\langle x, X \rangle^I} \diamond(\Box \neg A \wedge \Box \diamond \top) \wedge \diamond \diamond \Box \perp \end{aligned}$$

Proof. Suppose that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^*} \diamond(\diamond(\Box \neg A \wedge \Box \diamond \top) \wedge \diamond \diamond \Box \perp)$, and suppose for a reductio that $\mathcal{N}_{EK} \not\models_{\langle x, X \rangle^I} \diamond(\Box \neg A \wedge \Box \diamond \top) \wedge \diamond \diamond \Box \perp$. As the only other point R_{EK} -accessible to $\langle x, X \rangle^*$ is $\langle x, X \rangle^+$ this means that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^+} \diamond(\Box \neg A \wedge \Box \diamond \top) \wedge \diamond \diamond \Box \perp$. In particular it follows that that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^+} \diamond \diamond \Box \perp$. So there is a point $y \in R_{EK}(\langle x, X \rangle^+)$ such that $\mathcal{N}_{EK} \models_y \diamond \Box \perp$. So for some z such that $R_{EK}yz$, $\mathcal{N}_{EK} \models_z \Box \perp$. But this is impossible, since the only z for which $R_{EK}yz$ is $\langle y, Y \rangle^*$ for some neighborhood $Y \in N(y)$, and $\langle y, Y \rangle^*$ is not a point lacking R_{EK} -successors (in fact, having precisely two, namely $\langle y, Y \rangle^+$ and $\langle y, Y \rangle^I$). So by reductio it follows that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^I} \diamond(\Box \neg A \wedge \Box \diamond \top) \wedge \diamond \diamond \Box \perp$. \square

Lemma 4. For all formulas A and all $x \in W$ and $X \in N(x)$ we have the following.

$$\mathcal{N}_{EK} \models_{\langle x, X \rangle^I} \diamond(\Box \neg A \wedge \Box \diamond \top) \wedge \diamond \diamond \Box \perp \Rightarrow \mathcal{N}_{EK} \models_{\langle x, X \rangle^-} \Box \neg A \wedge \Box \diamond \top.$$

Proof. We begin by noting that $\diamond(\Box \neg A \wedge \Box \diamond \top) \wedge \diamond \diamond \Box \perp$ is a formula of the form $\diamond A \wedge \diamond B$ (with $A = \Box \neg A \wedge \Box \diamond \top$ and $B = \diamond \Box \perp$), and that we can't have $\diamond(A \wedge B)$ true at $\langle x, X \rangle^I$ when A and B are understood as above – as this would mean that $\Box \neg A \wedge \Box \diamond \top \wedge \diamond \Box \perp$, as this would mean that $\Box \diamond \top \wedge \neg \Box \diamond \top$ were true at one of $\langle x, X \rangle_{e_1}^I$ or $\langle x, X \rangle^-$. Thus we know that either (a) $\mathcal{N}_{EK} \models_{\langle x, X \rangle^-} \Box \neg A \wedge \Box \diamond \top$. or (b) $\mathcal{N}_{EK} \models_{\langle x, X \rangle_{e_1}^I} \Box \neg A \wedge \Box \diamond \top$. For (b) this would mean that $\mathcal{N}_{EK} \models_{\langle x, X \rangle_{e_2}^I} \diamond \top$ which is clearly not so, and thus it follows that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^-} \Box \neg A \wedge \Box \diamond \top$. \square

Lemma 5. For all points $x \in W$ and $X \in N(x)$ we have the following.

$$\mathcal{N}_{EK} \models_{\langle x, X \rangle^+} \Box \Box \diamond \top.$$

Proof. Suppose for a reductio that $\mathcal{N}_{EK} \not\models_{\langle x, X \rangle^+} \Box\Box\Diamond\top$. Then there must be a point $y \in R_{EK}(\langle x, X \rangle^+)$ and a point $z \in R_{EK}(y)$ such that $\mathcal{N}_{EK} \not\models_z \Diamond\top$. It is easy to see that such a point z must be of the form $\langle y, Y \rangle^*$ for some $Y \in N(y)$ and that such a point has exactly two R_{EK} -successors – namely $\langle y, Y \rangle^+$ and $\langle y, Y \rangle^I$ – and thus $\mathcal{N}_{EK} \models_z \Diamond\top$, giving us a contradiction. Thus by reductio it follows that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^+} \Box\Box\Diamond\top$ as desired. \square

Lemma 6. *For all points $x \in W$ and $X \in N(x)$ we have the following.*

$$\mathcal{N}_{EK} \models_{\langle x, X \rangle^-} \Box\Diamond\top$$

Proof. Suppose for a reductio that $\mathcal{N}_{EK} \not\models_{\langle x, X \rangle^-} \Box\Diamond\top$. Then there must be a point $y \in R_{EK}(\langle x, X \rangle^-)$ such that $\mathcal{N}_{EK} \not\models_y \Diamond\top$. But, as \mathcal{N} is a model with non-empty neighborhoods it follows that there must be at least some set of points Y such that $Y \in N(y)$. Thus, by the construction of \mathcal{N}_{EK} it follows that $Ry\langle y, Y \rangle^*$ – giving us a contradiction. Thus it follows that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^-} \Box\Diamond\top$ as desired. \square

Theorem 5. *Let $\mathcal{N} = \langle W, N, V \rangle$ be a neighborhood model with non-empty neighborhoods and $\mathcal{N}_{EK} = \langle W_{EK}, R_{EK}, V_{EK} \rangle$ be the model given by Definition 4. Then, for all formulas A and all points $x \in W$ we have the following.*

$$\mathcal{N} \models_x A \text{ if and only if } \mathcal{N}_{EK} \models_x \tau_{\Box'}(A).$$

Proof. By induction upon the complexity of A , the only case of interest being that in the inductive step where $A = \Box B$ for some formula B .

For the ‘only if’ direction suppose that $\mathcal{N} \models_x \Box B$. Then for $X = \|\Box B\|$, we have that $X \in N(x)$. By the inductive hypothesis it follows that for all points $y \in X$ that $\mathcal{N}_{EK} \models_y \tau_{\Box'}(B)$. By the definition of R_{EK} it follows that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^+} \Box\tau_{\Box'}(B)$ and by Lemma 5 $\mathcal{N}_{EK} \models_{\langle x, X \rangle^+} \Box\Box\Diamond\top$, and consequently that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^*} \Diamond(\Box\tau_{\Box'}(B) \wedge \Box\Box\Diamond\top)$. As the points $y \in X$ are the only points in \mathcal{N} where B is true we know that all the points $z \in W \setminus X$ are such that $\mathcal{N} \not\models_z B$ and so by the inductive hypothesis we know that, for all such points z $\mathcal{N}_{EK} \not\models_z \tau_{\Box'}(B)$ – and hence that $\mathcal{N}_{EK} \models_z \neg\tau_{\Box'}(B)$. By the definition of R_{EK} it follows that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^-} \Box\neg\tau_{\Box'}(B)$, and thus by Lemma 6 $\mathcal{N}_{EK} \models_{\langle x, X \rangle^-} \Box\neg\tau_{\Box'}(B) \wedge \Diamond\Box\top$. Thus it follows that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^I} \Diamond(\Box\neg\tau_{\Box'}(B) \wedge \Diamond\Box\top) \wedge \Diamond\Diamond\Box\perp$. Consequently we can see that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^*} \Diamond(\Box\tau_{\Box'}(B) \wedge \Box\Box\Diamond\top) \wedge \Diamond(\Diamond(\Box\neg\tau_{\Box'}(B) \wedge \Diamond\Box\top) \wedge \Diamond\Diamond\Box\perp)$ and thus that $\mathcal{N}_{EK} \models_x \tau_{\Box'}(\Box B)$.

For the ‘if’ direction suppose that $\mathcal{N}_{EK} \models_x \tau_{\Box'}(\Box B)$. Then by the definition of truth we have that there exists a point $y \in R_{EK}(x)$ such that $\mathcal{N}_{EK} \models_y \Diamond(\Box\tau_{\Box'}(B) \wedge \Box\Box\Diamond\top) \wedge \Diamond(\Diamond(\Box\neg\tau_{\Box'}(B) \wedge \Diamond\Box\top) \wedge \Diamond\Diamond\Box\perp)$. From the construction of \mathcal{N}_{EK} we know that such a y will be $\langle x, X \rangle^*$ for some $X \in N(x)$. By Lemma 2 it follows that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^+} \Box\tau_{\Box'}(B) \wedge \Box\Box\Diamond\top$. Thus for all points $y \in R_{EK}(\langle x, X \rangle^+)$ we have $\mathcal{N}_{EK} \models_y \tau_{\Box'}(B)$. By the inductive hypothesis $\mathcal{N} \models_y B$ for all such points y – and thus $X \subseteq \|\Box B\|$. By Lemma 3 it follows that $\mathcal{N}_{EK} \models_{\langle x, X \rangle^I} \Diamond(\Box\neg\tau_{\Box'}(B) \wedge \Diamond\Box\top) \wedge \Diamond\Diamond\Box\perp$, and thus by Lemma

4 $\mathcal{N}_{EK} \models_{\langle x, X \rangle} (\Box \neg \tau_{\Box'}(B) \wedge \Diamond \Box \top)$. Thus for all points $z \in W \setminus X$ we have that $\mathcal{N}_{EK} \not\models_z \tau_{\Box'}(B)$. By the inductive hypothesis $\mathcal{N} \not\models_z B$ for all such points z – and thus $W \setminus X = W \setminus \|B\|$. It follows then, that $X = \|B\|$ and thus that $\mathcal{N} \models_x \Box B$ as desired. \square

Theorem 6. *For all formulas A we have the following.*

$$A \in \mathbf{E} \text{ if and only if } \tau_{\Box'}(A) \in \mathbf{K}.$$

Proof. The ‘only if’ direction is Lemma 1. For the ‘if’ direction suppose that $A \notin \mathbf{E}$. Then by Theorem 3 there is a neighborhood model with non-empty neighborhoods $\mathcal{N} = \langle W, N, V \rangle$ and a point $x \in W$ such that $\mathcal{N} \not\models_x A$. By Theorem 5 it follows that $\mathcal{N}_{EK} \not\models_x \tau_{\Box'}(A)$ and thus, as this is a model on a Kripke frame, that $\tau_{\Box'}(A) \notin \mathbf{K}$ as desired. \square

As compared to the translation in section 2 our translation maps formulas of modal degree n to formulas of modal degree $5n$, while the translation derived from the Thomason translation maps such a formula to one of modal degree $7n$.

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