A Dialogical Route to Logical Pluralism

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Abstract

This paper argues that adopting a particular dialogical account of logical consequence quite directly gives rise to an interesting form of logical pluralism, the form of pluralism in question arising out of the requirement that deductive proofs be explanatory.

1 The Explanatory Problem for Logical Pluralism

Is there a unique answer to the question of whether a given argument is (deductively) valid? According to *logical monists* such as Read (2006) and Priest (2011), who hold that there is a unique correct logic, the answer is an emphatic 'yes', validity judgements being determined by the deliverances of the 'one true logic'. According to *logical pluralists* such as Beall and Restall (2006) and Shapiro (2014), by contrast, the answer to this question is 'no'. The logical pluralist holds that there are, in some sense, multiple different correct logics which we can use to assess the validity of any given argument, and which may deliver incompatible validity judgements concerning said argument. Those who wish to defend a pluralist position face a kind of explanatory challenge posed elegantly by Rosanna Keefe in the following quote:

> A characterisation of a pluralist position needs to explain what it is to endorse all of the various consequence relations the pluralist accepts and how they relate to an intuitive notion of logical consequence. (Keefe, 2014, p.1376)

So a logical pluralist needs to explain two things: firstly, how there can be more than one correct notion of logical consequence; and secondly, why these various different notions of logical consequence are notions of *logical* consequence. In order to meet this challenge one has to say something substantial about the nature of logical consequence. For example, in Beall and Restall (2006), the locus classicus on logical pluralism, logical consequence is defined in terms of what they call the 'Generalized Tarski Schema', according to which an argument is valid$_x$ iff the conclusion is true in every case$_x$ in which the premises are true. We then get different notions of validity by giving different specifications of what a case is. For example, construing cases as 'situations' gives us a Relevant notion of validity. To endorse such a notion of validity is just to take the 'actual case' to be amongst its cases. We will not detain ourselves with an assessment of the adequacy of this response to the explanatory challenge, referring the interesting reader to the extended discussion in section 3 of Keefe (2014) where this point is taken up. Our reasons for mentioning this particular account is merely to give the reader an example of what a response to this challenge might look like.

What we will do here is to use the explanatory challenge to enhance our understanding of a different, dialogical, account of logical consequence. In particular I will show how, properly understood, the Build-In Opponent conception of deduction naturally gives rise to an interesting form of logical pluralism (contingent on the existence of groups engaged in certain kinds of explanatory projects). We begin in Section 2 by giving an informal account of a dialogical conception of logical consequence, which we formalize in Section 3 in terms of a non-standard kind of dialogue game. In Section 4 I will then go on to argue that, properly understood, the Built-In Opponent conception of deduction gives rise to an interesting form of logical pluralism conditional on there being certain groups whose projects place particular kinds of constraints on what counts as an explanatory proof. Sections 4.1 and 4.2 then give examples of groups which, I argue, can be seen to be engaged in such explanatory projects. I conclude in Section 5 by examining the kind of pluralism which results from the above considerations. Formal results concerning the particular dialogue games we are concerned with are largely confined to a technical appendix at the end of the paper.

2 The Built-In Opponent Conception of Deduction

The Built-In Opponent (or BIO) conception of Deduction is a specific dialogical account of logic developed by Catarina Dutilh Novaes in a sequence of papers and applied to a number of issues ranging from the normativity of logic (Dutilh Novaes, 2015), the cognitive profile of deductive reasoning (Dutilh Novaes, 2013), reductio arguments (Dutilh Novaes, 2016), mathematical explanation (Dutilh Novaes, 2018), and the semantic paradoxes (Dutilh Novaes and French, 2018). According to this account, deductive arguments correspond to a specialized kind of semi-adversarial dialogues, involving both an adversarial as well as an importantly cooperative component. In these dialogues our two participants have opposed goals: the *Prover* (or Proponent) seeks to establish that a certain conclusion follows from given premises; while the *Skeptic* (or Opponent) seeks to block the establishment of this conclusion, typically by asking pointed 'why does this follow?' questions. We can think of these dialogues as involving the Skeptic asking for, and the Prover giving, reasons for their various claims of logical consequence (echoing the 'game of giving and asking for reasons' of Brandom (1994)). The adversarial nature of these dialogues is obvious, being a result of the opposed goals of the two players. At a higher-level, though, there is also a large degree of cooperation between the two players, as Dutilh Novaes explains:

> Proponent's job is not only to 'beat Opponent'; she also seeks to *persuade* Opponent of the truth of the conclusion, if he has granted the truth of the premises. In fact, the goal is not only to show *that* the conclusion follows from the premises, but also *why* it does; this corresponds to the idea that deductive arguments ought to have explanatory value. In this sense, Proponent and Opponent are cooperating in a

> common inquiry to establish what follows from the premises, and thus to further investigate the topic in question. (Dutilh Novaes, 2015, p.599f)

It is important to note that it is not only Prover's role which has a cooperative component. The Skeptic's role also requires a degree of higher-order cooperation, requiring them to not simply obstinately deny that a conclusion follows from some given premises, but instead to make clear to the Prover what would be required to convince them that the conclusion follows from the premises. One can see part of the tension between Achilles and the Tortoise in Carroll (1895) as arising out of the Tortoise not being a particularly cooperative Skeptic, and not being clear about what it would take to convince them that an inference via Modus Ponens is valid (resulting in them being a decidedly passive aggressive Skeptic).

Prover winning any particular dialogue with Skeptic, convincing them on that occasion that the conclusion follows from the premises given, does not necessarily meant that the argument in question is deductively valid, though. What is required for deductive validity is that Prover is able to win a dialogue with Skeptic no matter how Skeptic presses their case. This makes deductive validity correspond to the existence of a *winning strategy* for Prover, a strategy for engaging in a dialogue with Skeptic so that, no matter how Skeptic challenges their reasoning, Prover can give a response so that they eventually end up convincing them.

At a first glance deduction does not appear to involve the degree of multi-agent interaction which is present in the above picture. Typically we think of the act of determining whether an argument is deductively valid as being a rather solitary activity, involving at most one agent (and often barely even that!). This thought is at least partially correct. The other important aspect of the present approach is the idea that over time Skeptic has been progressively 'silenced' and idealized, until they are no longer an active participant in a deductive dialogue but have instead become part of the deductive method itself. In essence their role has been internalized and is played 'offline' by Prover. In order to beat such an idealized opponent Prover needs to ensure that there are no counterexamples to the inferential steps which they provide as responses to the Skeptic's queries. In later papers Dutilh Novaes explains the internalization of Skeptic in terms of what Pascual and Oakley (2017) call *fictive interaction*, noting that:

> Given the observation that a mathematical proof is a (specific, regimented and specialized) form of discourse, and the idea that interactive, conversational structures permeate much of what appears to be monological discourse at first sight, it is in fact not so surprising that a mathematical proof too would have a conversational (dialogical) basis. My main contribution here is to spell out in detail the kind of conversation that a mathematical proof re-enacts: semi-adversarial dialogues following fairly strict rules, involving the fictive participants Prover and Skeptic. (Dutilh Novaes, 2018, p.90)

This account is intended as an account of the *nature* of our deductive practices, of in some sense 'what is really going on' when we provide a deductively valid proof. We will not weigh in here on whether this is correct as an account of the nature or geneaology of our deductive practices, the view is at the very least both fruitful and interesting.

Rather what we will show here is that the way in which the Built-In Opponent conception of deduction highlights and emphasizes the connection between proofs and explanations points towards an interesting variety of logical pluralism.

3 Prover-Skeptic Games

In order to illustrate how the Built-In Opponent conception of deduction gives rise to a form of logical pluralism, it will be helpful to make use of an (admittedly rather simplistic) model of the kinds of dialogical interactions which are at the heart of this account. To this end we will make use of *Prover-Skeptic Games*, a variety of dialogue game introduced in (Sørensen and Urzyczyn, 2006, p.89–94), where they are used to game-theoretically characterize the implicational fragment of intuitionistic logic.[1] We will largely follow (and build on) the presentation of such games given in Shoham and Francez (2008), where they are used to characterize the (associative) Lambek calculus.[2] In order to get a feel for how these games work it will be helpful to consider (in Figure 1) the kind of situation which these games are meant to model.

The dialogue in Figure 1 has the structure of the kind of dialogical interactions which are at the heart of the built-in opponent conception of deduction. In particular we can see Scott playing the role of Skeptic, constantly asking Penelope (acting as Prover) to provide reasons for her assertions by cooperatively stating what would need to be shown in order to convince them, and Penelope acting as Prover providing such reasons. This is also the general structure of a Prover-Skeptic game. In Sørensen and Urzyczyn (2006) these are defined in rather austere syntactical terms, instead we will find it much more helpful to follow Shoham and Francez (2008) and define our Prover-Skeptic games in terms of an antecedently given natural deduction system. Moreover, we will also find it helpful to (following Shoham and Francez (2008)) characterize the moves in our game in terms of *sequents* rather than formulas, as the reader will see below.

3.1 Formal Preliminaries

For the sake of simplicity we will stick here with the language $\mathcal{L}$ of implicational logic,[3] the formulas of which are constructed out of a countable supply of propositional variables $p_0, p_1, p_2, \ldots$ using the binary connective '$\to$' of implication (i.e. p_i is a formula

[1]These games, like most dialogical approaches to logic, draw their inspiration from work done by Paul Lorenzen and Kuno Lorenz on dialogical logic (Lorenzen and Lorenz (1978)). As pointed out in Uckelman et al. (2014), post-Lorenzen work on dialogical logic can be seen as being split into two 'schools'—one focusing more on individual dialogues, the other on winning strategies. Developments in the former of these two schools are nicely summarized in Keiff (2009). The latter school takes Felscher (1985) as its initial impetus, with recent work including Sørensen and Urzyczyn (2007), Fermüller (2008), Alma et al. (2011), Uckelman et al. (2014), and Urzyczyn (2016).

[2]From a formal point of view the present paper 'fills in the gaps' between these two proposals, showing how to characterize the remaining implicational logics between intuitionistic logic and the associative Lambek calculus. For the sake of simplicity we only consider logics which have the structural rule of exchange.

[3]Readers interested in seeing what Prover-Skeptic games for a more involved language look like should consult Urzyczyn (2016), which gives a detailed account of what can be done with Prover-Skeptic style game semantics for Intuitionistic propositional and predicate logic.

*In the library two logic students, **Penelope** and **Scott** are arguing over the validity of the argument from $p \to q$ and $q \to r$ to $p \to r$. Penelope thinks that this argument is valid and is trying to convince Scott, who is skeptical.*

- **Penelope (1)**: I reckon that $p \to r$ follows from $p \to q$ and $q \to r$
- **Scott (1)**: Yeah? Well if that's so then suppose I grant you $p \to q$ and $q \to r$ along with p, how are you meant to get r?
- **Penelope (2)**: If you grant me q I can get r from $q \to r$ (which you just granted).
- **Scott (2)**: But why should I grant you q?
- **Penelope (3)**: Well if you were to grant me p then I could get q from $p \to q$ which you granted at the start.
- **Scott (3)**: But why should I grant you p?
- **Penelope (4)**: Because you granted it to me at the start!

Penelope leaves the library triumphantly.

Figure 1: **A Dialogue** representing the kind of situation modelled by Prover-Skeptic games.

of $\mathcal{L}$ for all $i \in Nat$, and if A and B are formulas then so is $A \to B$).[4] A *sequent* is a pair $\langle \Gamma, A \rangle$ of a multiset Γ of formulas, and a formula A, which we will write throughout as $\Gamma \succ A$. Recall that a multiset is like a set, except that elements can occur multiple times so, for example $[A, A, B]$ and $[A, B]$ are distinct multisets.[5]

Throughout we will be concerned with fragments of the following natural deduction system, which result from dropping some or all of the structural rules listed.

AXIOMS

$$A \succ A$$

INTRODUCTION/ELIMINATION RULES

$$\frac{\Gamma, A \succ B}{\Gamma \succ A \to B}\ (\to I) \qquad \frac{\Gamma \succ A \quad \Delta \succ A \to B}{\Gamma, \Delta \succ B}\ (\to E)$$

STRUCTURAL RULES

$$\frac{\Gamma, A, A \succ B}{\Gamma, A \succ B}\ (W) \qquad \frac{\Gamma \succ B}{\Gamma, A \succ B}\ (K)$$

Natural deduction systems are rarely given in this 'structurally explicit' sequent-to-sequent style, notable exceptions being the natural deduction system for intuitionistic

[4] Throughout we use uppercase Roman letters A, B, C, D,etc (with and without subscripts) as schematic letters for formulas of $\mathcal{L}$, uppercase Greek letters as schematic for multisets of formulas/sequents, and lowercase Greek letters as schematic for sequents.

[5] For more information on multisets and their history see Singh et al. (2008).

logic given in (Dummett, 1977, p.88f) which has an explicit rule of weakening (=(K) above), or the system Nat in (Humberstone, 2011, p.114), in which it is shown that weakening (there called 𝕄) is derivable given the rules for conjunction. In both of these cases the need for an explicit rule of contraction is avoided by working with sets rather than multisets. Usually structural variation is captured in natural deduction systems in terms of different policies regarding how assumptions can be discharged, but we will find it particularly helpful to be fully explicit about the applications of structural rules throughout.

3.2 Defining the Game

In order to streamline our description of Prover-Skeptic games it will be helpful to have a few definitions on the table first. Given a multiset of sequents S and sequents β, γ let us write $\beta \Rightarrow^*_{(I)} \gamma$ whenever we can derive γ from β using zero or more applications of our introduction rules (in our case $(\rightarrow I)$), and $S \Rightarrow^*_{(E)} \beta$ iff we can derive β from the sequents in S via zero or more applications of our elimination and structural rules (i.e. using zero or more applications of $(\rightarrow E)$, (W) and (K)). Finally we will say that a sequent $\Gamma \succ A$ is *atom-focused* whenever A is a propositional variable.

Definition 3.1. *(Dialogue Game) A* dialogue *over* (Γ, A) *is a (possibly infinite) sequence* $S_1, \beta_1, S_2, \beta_2, \ldots$ *where each* S_i *is a multiset of sequents, and each* β_i *a sequent where:*

1. $S_1 = [\Gamma \succ A]$
2. $\beta_i \Rightarrow^*_{(I)} \sigma$ *for some non-axiomatic* $\sigma \in S_i$ *and some atom-focused sequent* β_i.
3. $S_{i+1} \Rightarrow^*_{(E)} \beta_i$

In actual (informal) dialogues, we have participants asserting, denying, and querying sentences, while here the corresponding sequents register the discursive commitments incurred by a particular participant and what follows from those commitments. We can read a sequent $\Gamma \succ A$ as claiming that 'you cannot grant me Γ without also granting me A'. If we read sequents in this way, then we can understand Prover Skeptic games as defined above as having the following form.

- The game begins by Prover offering the sequent $\Gamma \succ A$, claiming that if they are granted Γ then they must also be granted A.
- A *Skeptic* move involves them challenging some sequent (σ) offered by Prover in their previous move, with Skeptic offering a sequent (β) of the form $\Gamma \succ p$ that entails the challenged Prover sequent. In offering the sequent $\Gamma \succ p$ the Skeptic is essentially saying 'if you demonstrate to me that $\Gamma \succ p$ then I'll grant you that σ is the case'.
- A *Prover* move in response to a Skeptic challenge (β) involves them offering a set of sequents (S) from which they claim they can derive the Skeptic challenge (β) using only elimination rules and the structural rules, and hence (given that the

Skeptic challenge β entails the sequent σ which they challenged) the sequent initially challenged by Skeptic.[6]

What is not clear from the above description is how it is that the game ends, and this is determined by the constraints placed on the structure of Skeptic moves. A Skeptic cannot challenge just any sequent offered by Prover, though, some challenges are ruled out by their previous discursive commitments. In particular, because of the structure of our dialogues, a Skeptic can never challenge a sequent that is an instance of (Id), as, in order for Prover to legally offer a sequent of the form $A \succ A$ Skeptic will have previously had to have granted A. In this way, then, we will say that *Prover wins a dialogue* whenever there is no sequent which Skeptic can legally challenge, i.e. when Prover is able to offer only instances of (Id).[7]

Winning a dialogue does not mean that the sequent initially offered by Prover is valid, though—it could be that Skeptic simply challenged the wrong sequents, and had they played differently then there would have been no way for Prover to win. Instead, what is needed in order for us to be able to ascertain that a sequent is valid is for Prover to have a *winning strategy*—for there to always be a move which Prover can make in response to any Skeptic challenge so that they eventually win.

Definition 3.2. *(Winning Strategy) A* winning strategy *(for Prover) for the dialogue over* (Γ, A) *is a tree where each node on each branch is labelled alternatingly with either P or S, where:*

- *The root note of the tree is a P-node* $([\Gamma \succ A])$.
- *Each branch is a dialogue over* (Γ, A)
- *Every P-node* S_i *has an S-node child* β_i *for each non-axiomatic sequent* $\sigma \in S_i$*, where* $\beta_i \Rightarrow^*_{(I)} \sigma$.
- *Every S-node has a single P-node child.*
- *All branches are finite and end with a set* S *of instances of* (Id).

In the appendix we show that if Prover has a winning strategy for the dialogue over Γ, A then $\Gamma \succ A$ is provable in the proof system given above.

In order to make it slightly more concrete how these games work let us consider an example. In this, and all subsequent examples, we will label each move, for ease of reading, with the labels 'P:' for 'Prover' and 'S:' for 'Skeptic', listing the contents of the move after the colon. Throughout all our examples we will separate items in our multisets using a semi-colon rather than a comma for ease of readability.

To see how everything works let us return to the dialogue between Penelope and Scott given in Figure 1 to informally motivate the structure of Prover-Skeptic games. This is a dialogue over $([p \to q, q \to r], p \to r)$, and proceeds as follows.

[6] It is worth noting that we could be significantly more restrictive in what a Prover move looks like without any loss of available winning strategies for Prover. For example, we could enforce conditions similar to those laid down in (Shoham and Francez, 2008, p.175f) which force Prover to explicitly use a sentence granted by Skeptic in order to derive the challenge. We will stick with the current definition for the sake of perspicuity, leaving a fuller discussion of this issue to the Appendix.

[7] See (Dutilh Novaes and French, 2018, p.149) for an example of where this situation needs to be made slightly more nuanced.

P: $[p \to q, q \to r \succ p \to r]$

Prover begins by stating the claim they wish to defend.

S: $p \to q, q \to r, p \succ r$

Skeptic responds by challenging the only available sequent, requesting that Prover demonstrate how they can get here.

P: $[q \to r \succ q \to r \; ; \; p \to q, p \succ q]$

Prover responds by providing a collection of sequents from which they can derive the challenge via $(\to E)$.

S: $p \to q, p \succ q$

Skeptic challenges the only non-axiomatic sequent in the previous Prover move.

P: $[p \to q \succ p \to q \; ; \; p \succ p]$

Prover replies offering two axiomatic sequents.

S: No Available Move

As all of the sequents which Prover offered are axioms there are no further moves available to the Skeptic.

Given that at no stage could Skeptic have made any other move, the above dialogue corresponds quite directly to a winning strategy for Prover.

3.3 Internalization & Cooperation

Any successful formalization of the built-in opponent conception of deduction must be able to explain how it is that Skeptic is apt to be internalized, ideally shedding some light in the process on why it is Skeptic's role which has become internalized rather than Prover's. As it turns out, Prover-Skeptic games give quite an elegant account of why this is the case. As can be seen in the above example, Prover has quite a lot of freedom in determining which sequents to offer as part of their move, in fact in the above presentation Prover is completely unconstrained in which sequents they can offer so long as they can be used to derive the challenge using only elimination and structural rules. As it turns out we could also constrain Prover in various ways without altering which sequents Prover has winning strategies for (for example, by forcing them to either offer a singleton multiset (and derive the challenge via structural rules) or in replying to a skeptic move $\Gamma \succ A$ to offer a multiset of sequent which includes the sequent $B \succ B$ for some $B \in \Gamma$. Even in these cases Prover has quite a lot of leeway in determining which sequents to offer. By contrast, once they have determined which sequent they are going to challenge, Skeptic's move is completely deterministic in the sense illustrated by the following simple lemma.

Lemma 3.3. *Suppose that $\sigma = \Gamma \succ A$, where:*

- $A = B_1 \to (\ldots(B_{n-1} \to (B_n \to p_i))\ldots)$
- $\Gamma' = [B_1, \ldots, B_n]$.

Then for all atom-focused sequents β_i*, we have* $\beta_i \Rightarrow^*_{(I)} \sigma$ *iff* $\beta_i = \Gamma', \Gamma \succ p_i$.

This means that Skeptic's moves do not require any form of creativity to perform. Given that their role is essentially algorithmic it is easy to see how it is apt to be performed 'offline' by a Prover who follows this method.

The other main feature which a formalization of the built-in opponent approach would have to explicate is the mix of higher-level cooperation and lower-level adversariality between the two players. The adversariality is present here due to the conflicting goals of the two players. The higher-level cooperation can be seen not just in the fact that the two players are assumed to follow the rules (that variety of cooperation is, in this setting, largely uninteresting), but also in the structure of Prover and Skeptic moves. In particular Skeptic doesn't simply object to a sequent put forward by Prover and exclaim 'that's invalid!'. Instead they make it clear to Prover what Prover must do in order to convince them. Similarly, Prover doesn't just put forward any random collection of sequents, but instead some from which the challenge sequent could be derived. We could even enforce further cooperation between Prover and Skeptic by placing the further restrictions mentioned above on the structure of Prover moves. In talking about the importance of the mix of higher-level cooperation and lower-level adversariality Catarina Dutilh Novaes says the following

> The adversarial component is what accounts for the property of necessary truth preservation; the cooperative component is what explains the ideal of perspicuity and explanatoriness. (Dutilh Novaes, 2015, p.599f)

So not only are adversarial and cooperative components present in Prover-Skeptic games, they appear to be present *in the right way*. Furthermore, as we will argue in the next section, it also helps to make clear how proofs can have explanatory value.

4 A Route to Pluralism

Prover moves are are meant to do more than merely show that the premises follow from the conclusion, they're meant to be *explanatory*. The notion of explanation we have in mind here is something broadly 'pragmatic' in nature, one which follows Van Fraassen (1988) in thinking of explanations as involving the answers to why-questions. Understanding explanation as having this feature allows us to show how Prover moves are meant to be explanatory: Prover moves are answers to the question 'why does this follow'? A why-question is a request for information of a certain sort and, as shown in (Van Fraassen, 1988, p.153), what information is requested can differ from context to context. This audience-relativity of explanations is seen as a central aspect of the explanatoriness of proofs when viewed from the BIO perspective.

> The main feature of the dialogical account of explanatoriness in mathematical proofs defended here is that explanatoriness is 'in the eyes of the beholder': it is not a property of a proof as such, but rather it emerges from the *relation* between a proof and a given audience. (Dutilh Novaes, 2018, p.91)

So according to the BIO conception of deduction what counts as an adequate response to a Skeptic challenge will be relative to a particular community of inquiry, the putative intended audience of the proof. As we will see, it is the way in which what counts as an explanation is relative to the interests and ends of particular communities which potentially provides a route to an interesting form of logical pluralism.

Everything we have said above is perfectly compatible with logical monism, nothing that we have said so far entails that the different constraints which are actually placed on what counts as a response to a Skeptic challenge by actual communities will have any repurcussions about which arguments are deductively valid. Indeed, many of the different constraints placed on what counts as an adequate response to a Skeptic challenge will not affect validity judgements, but instead concern issues pertaining to the level of detail required in, for example, teaching situations as opposed to research mathematics (Hersh (1993)).[8] By and large concerns about granularity and 'level of detail' are absent from the present account, although one can see some remnant of it in the following example. Suppose we are at a stage in a dialogue where the previous Skeptic challenge is as follows.

S: $p \to (q \to r), p \to q, p, p \succ r$

In such a situation Prover might reply in at least the following two ways:

P: $[p \succ p\,;\; p \to q \succ p \to q\,;\; p \to (q \to r) \succ p \to (q \to r)\,;\; p \succ p]$

P: $[p, p \to q \succ q\,;\; p \to (q \to r) \succ p \to (q \to r)\,;\; p \succ p]$

Both of these are multisets of sequents from which one can derive $p \to (q \to r), p \to q, p, p \succ r$ simply using $(\to E)$, but the second of these provides more details as to how this is to be done in the sense that only two applications of $(\to E)$ are required to derive the challenge, while in the first case three are required. A situation where the norms on explanation disallowed the first prover move, but not the second, would result merely in there being fewer winning strategies for some sequents, not in there being fewer sequents which have winning strategies.

This raises an important question: are there contexts/communities whose explanatory projects place constraints on what counts as an adequate response to a Skeptic challenge which have repercussions for what arguments are valid? To the extent to which the answer to this question is 'yes' we can see an interesting form of logical pluralism arising out of the Built-in Opponent conception of deduction according to which, contra (Beall and Restall, 2006, p.88), what arguments count as valid are relative to communities of inquiry, and their practical interests. What I will do in the rest of the paper is first give some reasons for thinking that the answer here might indeed be 'yes', and then examine the resulting variety of logical pluralism—addressing the question of whether it is indeed a form of logical pluralism at all!

[8] Hersh claims that there are two different roles which proofs play in these two contexts (explaining in teaching situations, and convincing in research ones). The differences which Hersh points to, though, are largely ones concerning granularity and level of detail.

4.1 Case 1: Geometry and Contraction

We will begin by looking at what is both the most interesting, and also the most contentious, potential example of a community in which the norms on explanations result in different inferences being valid. This case is drawn from mathematicians working in the foundations of geometry at the beginning of the 20th century. In particular, Pambuccian (2004) points out that mathematicians in that period were concerned with not just which axioms are used in proving a particular theorem, but also with the *number of times* such axioms are, or must, be appealed to in proving such a theorem. For example, G. Hessenberg in 1905 proves that Desargues axiom follows from a collection of axioms for plane projective geometry along with a threefold use of the Pappus axiom. Pambuccian notes that all known proofs both require, and explicitly mention, this threefold use. This is precisely the kind of situation which Mares and Paoli claim put pressure on the validity of structural contraction:

> "It makes perfect sense to say, for example, that a given mathematical theorem T 'follows from' the axioms of the theory it belongs to. Are we thereby claiming that all the axioms of the theory have been *used* to establish it, or even that they are relevant in content to it? No, of course: only that whenever we grant the axioms themselves, we are committed to accept T. This is a sense of 'follows from' for which weakening and contraction seem beyond doubt. But it does not appear to be the same meaning that we have in mind when claiming that T 'follows from', say, a lemma L. For us to appropriately affirm that, it seems necessary that L be at least *used* in the proof of T.
>
> Yet, there is more to it. Suppose you have a proof of T that requires two applications of the axiom of choice and that we challenge you to find a proof where the same axiom is applied only *once*. You may or may not succeed–if you don't, perhaps you will start wondering whether Structural Contraction is adequate for this particular sense of 'follows from'."(Mares and Paoli, 2014, p.452f)

Here we can see Mares and Paoli offering an argument in favour of rejecting contraction in precisely cases like this one. According to the present account this suggests that what we can see here is a change in what counts as an explanatory proof—i.e. a change in what counts as an adequate response to a Skeptic challenge—arising out of the foundational interests of turn of the century geometers. We need not give some account of *why* it is that geometers at the turn of the 20th century appear to have been so concerned with the number of times axioms are appealed to in proving theorems, only that they did seem to place such a constraint on what counts as a suitably explanatory proof.[9]

So it appears that here we have a community in which it is not sufficient that you show that the challenged claim follows merely 'of necessity' from the claims provided, but that you must also indicate the extent to which different claims are used in showing this, a community in which explanatory proofs need to not only show *that* a given conclusion follows from some premises, but also *how many copies* of those premises it follows from.

[9] It should be noted, of course, that this is not the only reconstruction of the situation here which is possible. Pambuccian himself favours one according to which this concern with the number of times a given axiom must be appealed to as reflecting a concern with the 'fine points of proof-theory'.

4.2 Case 2: Hypothetico-Deductivism, Confirmation, and Weakening

Our other primary case study concerns the Hypothetico-Deductive model of theory confirmation. Hypothetico-Deductivism is a method for confirming hypotheses by deriving novel predictions and then verifying them. In particular it involves taking a theory T and a hypothesis H and then deriving some prediction e from T and H which cannot be derived from T alone. According to Glymour (1980) (as summarized by Waters (1987, p.453)) Hypothetico-Deductivism is "hopeless as a method of confirmation because it cannot maintain the condition of relevance between a hypothesis and the evidence which confirms it". What is pointed out particularly clearly in Waters (1987) is that Hypothetico-Deductivism can be saved if we formulate it in terms of a Relevant logic à la Anderson and Belnap (1975) (related suggestions are also made in Schurz (1991) and Gemes (1993)).

In particular Glymour's complaint can be seen as suggesting that when we are dealing with theory confirmation, there needs to be an appropriate connection of relevance between theory, hypothesis and evidence in order for a given proof to show that some evidence confirms a hypothesis. This means that in contexts where we are attempting to explain why the combination of a given theory with a hypothesis we are trying to confirm derives a novel prediction, we will have to refrain from using the structural rule of weakening. Otherwise, we will be able to move from seemingly genuine cases of confirmation (for example that the Stark Effect confirms quantum mechanics) to spurious ones (like the Stark Effect confirms quantum mechanics and the infallibility of the Pope).

So for a Prover move to be sufficiently explanatory in the case where we're attempting to show that a given hypothesis is confirmed by some evidence it must *use* all of the sentences which are granted by Skeptic in order to derive the Skeptic's challenge. So it seems like in cases where our proofs are aimed at an audience with the goal of showing that evidence confirms a hypothesis that, following Waters (1987), we are in a situation where appropriate Prover moves are constrained in such a way as to not involve weakening.[10]

5 Explanations and Pluralism

The picture that emerges out of all of this is as follows. Whether an argument is valid or not depends on whether Prover has a winning strategy in a Prover-Skeptic game, where each of their moves are required to be not only truth-preserving, but also suitably explanatory. That is to say, Prover has to not only show *that* the argument is valid, but also show *why* the argument is valid. What we have canvassed above, though, are

[10]Of course, this does not allow us to escape all problems concerning confirmation (something which is not noted by Waters, but is noted by Routley and Nola (1991, p.17)). For example, as contraposition is still valid in the obvious way of extending this logic with negation (putting us in the vicinity of the $\{\to, \neg\}$-fragment of the relevant logic R) we will still run up against Hempel's paradox of the ravens. Of course, the move to thinking of confirmation in terms of a conditional which does not satisfy weakening allows us to deal with Hempel's paradox by modifying Nicod's criteria for instantial confirmation in the way discussed in Goddard (1977).

two potential situations where requirements on what counts as an explanation of why something follows results in certain arguments no longer being valid, not because the argument isn't truth preserving, but rather because it has no sufficiently explanatory proof. This results in a picture of logic on which what counts as valid is dependent on the audience that the argument is intended for—different communities of inquiry having different standards on what counts as an explanatory proof, and this can potentially have repercussions for which arguments have winning strategies.

It is important to note that these changes in what counts as an explanatory proof only affect Prover moves by restricting what structural rules they are able to appeal to in deriving the Skeptic's challenge. This is particularly important given that it is Prover's job to provide explanations to Skeptic as to why things follow. Let us look at an, admittedly simple and rather contrived, example taken from (Waters, 1987, p.459). Suppose we have a theory which draws a connection between Ravens and Blackness so that it claims that $\forall x(Rx \to Bx)$. Does such a theory claim that there is a connection between mammals having hair (S), and Ravens being black? Against a hypothetico-deductivist background, this amounts to the question of whether we can show that $\forall x(Rx \to Bx) \succ S \to \forall x(Rx \to Bx)$ is valid? Now, treating $\forall x(Rx \to Bx)$ as an atomic sentence, we have the following situation:

P(1): $[\forall x(Rx \to Bx) \succ S \to \forall x(Rx \to Bx)]$

S(1): $S, \forall x(Rx \to Bx) \succ \forall x(Rx \to Bx)$

If we are in an explanatory context in which the rule (K) of weakening cannot be applied then this is the best that Prover can do, and at this stage cannot offer any sequents to derive the Skeptic challenge.[11] If, instead, Prover was merely trying to convince Skeptic that this inference was truth preserving (an explanatory context in which weakening is certainly allowed), then they could reply to the Skeptic challenge S(1) with:

P(2): $[\forall x(Rx \to Bx) \succ \forall x(Rx \to Bx)]$

So we can see in this simple example how, depending on the explanatory context, what counts as an admissible Prover move can result in differences in which arguments are valid. In the appendix we show that if Prover has a winning strategy for a dialogue which makes use of some subset of the structural rules $\{(K), (W)\}$ then that sequent is derivable in our proof system using the same subset of structural rules.

So is this a kind of logical pluralism?[12] There is good reason for thinking so. What we have outlined above is a relativist picture of logic, according to which what counts as the correct account of validity is relative to what counts as an explanatory proof. This makes which logic is correct, given the pragmatic picture of explanation used here, relative to communities of inquiry and the explanatory projects they are engaged in. As noted in (Cook, 2010, p.493), for this to result in a pluralism about validity would

[11] Why? Because if Prover had a winning strategy for what in the Appendix we call a (W)-dialogue for $S, \forall x(Rx \to Bx) \succ \forall x(Rx \to Bx)$, then it would (by Proposition A.12) have a winning strategy in a strict (W)-dialogue, which it does not as there is no strict Prover responce that can be made in a (W)-dialogue at this point.

[12] I am indebted to an anonymous referee for pressing me on this point.

require there to be more than one relevant account of what counts as an explanatory proof *which makes a difference for which arguments are valid* (this second qualification is implicit in Cook's discussion).

Given that, as van Fraassen notes (Van Fraassen, 1988, p.143), what counts as an explanation can vary from context to context, it follows that on the account argued for here, whether an argument is valid can also vary from context to context. This is not so alien a thought, discussed extensively in Shapiro (2014), and advocated for in some detail by Caret (2017). Unlike Caret's approach, the present approach makes use of already existing features of contexts (those which are required to accommodate questions) in order to explain how whether an argument is valid can vary from context to context, and doesn't require the postulation of a separate 'deductive standard'.[13]

6 Conclusion

The Built-In Opponent conception of deduction gives an account of logical consequence in terms of dialogues where the contributions of both parties—Prover and Skeptic—are required to be suitably explanatory, Skeptic explaining what it would take to convince them, and Prover explaining why a give sequent is valid. But the kinds of explanations which Prover is required to give here are ones which are dependant on the projects of particular communities of inquiry. This opens up the possibility that what counts as valid may be dependant on what it takes to count as an appropriate explanation.

Whether the BIO conception of deduction gives rise to logical pluralism is an empirical matter, having to do with the explanatory projects of extant communities of inquiry. Above we've given two examples of communities of inquiry which illustrate how it may be that what counts as an explanation in different communities may have logical repercussions.

The resulting picture of the connections between logical consequence and the explanatory projects of different communities of inquiry allows for a natural answer to the explanatory challenge with we saw at the beginning of the paper. Recall that an answer to the explanatory challenge involves saying both what it is to endorse a consequence relation—explaining how there can be a plurality of correct logics—as well as saying how these different notions of logical consequence accord with an intuitive notion of logical consequence. Say that consequence relation *accords* with a constraint on explanations when an argument is valid according to that consequence relation iff Prover has a winning strategy in the Prover-Skeptic game where each of their moves abides by that constraint. Then to endorse a given consequence relation is to be engaged in an explanatory project which places constraints on explanation which accord with that consequence relation. If there are multiple different explanatory projects which accord with distinct consequence relations, then we have a genuine form of logical pluralism. Moreover, all of these different consequence relations relate to a single

[13] Furthermore, Caret proposes his approach as a way of defending Beall & Restall's version of pluralism against a variety of objections, while the present approach is proof theoretic in character, bearing a closer resemblance to the picture outlined in Restall (2014).

given, and hopefully intuitive, notion of logical consequence, namely that captured by the Built-In Opponent conception of deduction.

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Appendix: Adequacy Proofs

In this appendix I provide appropriate adequacy proofs for the kinds of Prover-Skeptic games which result from limiting which structural rules can be used by Prover in deriving the Skeptic challenge.

Definition A.1. *Let us call a dialogue D in which no Prover moves make use of contraction an* affine dialogue *(or (K)-*dialogue*), one in which no Prover move makes use of weakening a* relevant dialogue *(or (W)-*dialogue*), and one in which no Prover move makes use of either weakening or contraction a* linear dialogue. *Note that a dialogue is affine and relevant iff it is linear.*

Definition A.2. *Suppose that* S $\subseteq \{(W), (K)\}$. *Let us say that* $\vdash_S \Gamma \succ A$ *iff there is a proof of* $\Gamma \succ A$ *in which the only structural rules which appear are those in* S.

As it turns out the 'Completeness' direction of this proof is relatively simple due to the direct connection between Prover and Skeptic moves and derivations.

Proposition A.3 ('Completeness')**.** *Let* S $\subseteq \{(W), (K)\}$. *If there is a winning strategy for Prover over the* S*-dialogue* (Γ, A) *then* $\vdash_S \Gamma \succ A$.

Proof. Suppose that D is a winning strategy for the S-dialogue over (Γ, A). What we will show is that for any subtree D' of D whose root is a P-node (Σ) that all $\sigma \in \Sigma$ are S-derivable.

In the case where σ is an axiom this is trivial, so suppose that $\sigma \in \Sigma$ is not an axiom, and let β be the skeptic challenge for σ. As the strategy is a winning strategy it follows that the S-node β has a P-node descendant Σ'. By the induction hypothesis it follows that we have $\vdash_S \sigma'$ for all $\sigma' \in \Sigma'$. Furthermore, as each branch in the strategy must be a S-dialogue it follows that $\Sigma' \Rightarrow^*_{(E)} \beta$. So by pasting together derivations for each of the σ' appropriately we can, using the elimination rules and the structural rules S, construct a derivation for β, allowing us to conclude that $\vdash_S \beta$. But we know that $\beta \Rightarrow^*_{(I)} \sigma$ and so by applying successive introduction rules to β we can derive σ, allowing us to conclude $\vdash_S \sigma$ as desired. □

The more interesting direction is 'Soundness'. The main obstacle to soundness arises out of the way in which we are only limiting Prover's moves (and thus, considered proof-theoretically, only placing constraints on the E-sections of proofs). What we will now show is that we are nonetheless able to capture the above mentioned notions of S-consequence by showing that we can always permute the structural rules over instances of $(\rightarrow I)$ (Lemma A.9), and thus by making use of the normal form theorem, can transform every proof into one in which structural rules are only applied in the E-sections of a proof (Theorem A.10).

Definition A.4 (Normal Form (Prawitz, 1965))**.** *A proof is in* normal form *if no formula occurrence is both the consequence of the application of an* I*-rule, and the major premise of the application of an* E*-rule.*

Definition A.5. *A* stack *in a proof* π *is a sequence of sequents* $\Gamma_0 \succ A_0, \ldots, \Gamma_n \succ A_n$ *from* π *s.t.*

1. $\Gamma_0 = \{A_0\}$ *(i.e.* $\Gamma_0 \succ A_0$*) is a leaf sequent from* π*)*
2. $\Gamma_{i+1} \succ A_{i+1}$ *is immediately below* $\Gamma_i \succ A_i$ *in* π *for* $0 \leq i < n$.
3. *For* $0 \leq i < n$ *the sequent* $\Gamma_i \succ A_i$ *is not a minor premise of an application of* $(\rightarrow E)$.

A stack of order 0 *ends in the endsequent of* π*; a* stack of order $n+1$ *ends in the minor premise of an application of* $(\rightarrow E)$ *whose major premise belongs to a stack of order* n.

Our notion of a 'stack' of sequents is just the obvious lifting of the standard notion of a 'track ' of formula occurrences in a proof (on which cf. (Troelstra and Schwichtenberg, 2000, p.143)), but will allow us to be more explicit about our talk of the premises involved in tracks in proofs. Just like with a track, in a normal derivation it is easy to see that every stack of sequents can be split up into three parts: an E-part, followed by a 'minimum sequent', followed by an I-part. In the I-part all of the rules used are I-rules (or structural rules), in the E-part all the rules used are E-rules (or structural rules).

Normal proofs alone will not be sufficient for our proof below: we will need our normal proofs to satisfy two additional requirements concerning the 'minimal sequents' in stacks, and the applications of structural rules. In particular we will be interested in proofs in which the minimal sequent in each stack in a proof is of the form $\Gamma \succ p_i$ for some propositional variable p_i (call a proof in which this is the case one in *long normal form*), and in which no applications of structural rules appear in the I-part of a stack. As it happens though, every normal proof (and every proof whatsoever) can be transformed into one which both of these properties.

Proposition A.6. *Every proof in normal form can be transformed into one in long normal form.*

Proof. We can reduce the minimal formula in any track into one which is atomic by repeatedly applying the following reductions, each of which both reduce the complexity of minimal formula while still retaining the structure of the proof.

$$\begin{array}{c}\pi_0\\ \vdots\\ \Gamma \succ A \to B\\ \vdots\\ \pi_1\end{array} \quad\mapsto\quad \begin{array}{c}\pi_0\\ \vdots\\ \dfrac{\dfrac{\Gamma \succ A \to B \qquad A \succ A}{\Gamma, A \succ B}\ (\to E)}{\Gamma \succ A \to B}\ (\to I)\\ \vdots\\ \pi_1\end{array}$$

Note, also, that these reductions also preserve the rank of tracks.

□

The lambda terms for proofs in normal form are in β-normal form, and those which are in long normal form are in what is sometimes called $\beta\eta$-long normal form or $\beta\eta^-$ normal form.

Definition A.7 (Top-Heavy Normal Form)**.** *Say that a proof π is in* top-heavy normal form *if it is in long normal form and no applications of structural rules appear in any I-part of the proof.*

Theorem A.8 (Normal Form Theorem (Prawitz, 1965))**.** *(i) If there is a proof of $\Gamma \succ A$, then there is a normal proof of $\Gamma \succ A$; (ii) If there is a proof of $\Gamma \succ A$, then there is a proof of $\Gamma \succ A$ in long normal form.*

Lemma A.9 (Permuation of Structural Rules)**.** *Suppose that π is a (normal) proof. Then there is an equivalent (normal) proof of the same height and of the same endsequent in which all occurrences of structural rules appear only after applications of $(\to E)$ or axioms.*

Proof. Proof is by induction on the number of structural rules which appear after any given instance of $(\to I)$. If that number is zero, then we are done. Otherwise we have a situation like the one on the ones on the left in the following diagram, in which case we can transform the proof to one on the right, 'pushing' the structural rule over the instance of $(\to I)$.

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\\ \Gamma, A \succ B\end{matrix}}{\Gamma \succ A \to B}\,{\to}I}{\Gamma, C \succ A \to B}\,(K) \quad\mapsto\quad \dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\\ \Gamma, A \succ B\end{matrix}}{\Gamma, A, C \succ B}\,(K)}{\Gamma, C \succ A \to B}\,{\to}I$$

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\\ \Gamma, C, C, A \succ B\end{matrix}}{\Gamma, C, C \succ A \to B}\,{\to}I}{\Gamma, C \succ A \to B}\,(W) \quad\mapsto\quad \dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\\ \Gamma, C, C, A \succ B\end{matrix}}{\Gamma, C, A \succ B}\,(W)}{\Gamma, C \succ A \to B}\,{\to}I$$

Repeatedly doing this will result in all instances of structural rules appearing directly after applications of modus ponens or axioms. □

Theorem A.10. *If a sequent $\Gamma \succ A$ is S-provable then it has a S-proof in top-heavy normal form.*

Proof. If the sequent is provable then by Theorem A.8 it has a proof in long normal form. By Lemma A.9 we can permute all structural rules so that they appear only after applications of $(\to E)$. In any given thread, then, this will result in no structural rules being applied in the I-section of the proof, putting it in top-heavy normal form. □

Given a proof π let us say that a sequent $\Gamma \succ A$ in π is *challengeable* iff $\Gamma \succ A$ is not an axiom, and is the final sequent in a stack. Further, let us say that the *height above a challengeable sequent* $\Gamma \succ A$ is equal to the number of stacks of greater rank than the stack in which $\Gamma \succ A$ occurs.

Proposition A.11 ('Soundness'). *Let S $\subseteq \{(W), (K)\}$. If $\vdash_S \Gamma \succ C$ then there is a winning strategy for Prover over the S-dialogue (Γ, C).*

Proof. Suppose that $\vdash_S \Gamma \succ C$. Then by Theorem A.10 we know that it has a proof π in top-heavy normal form. What we will now do is use this proof to construct a winning strategy for the S-dialogue over (Γ, C).

We proceed by induction on the height above challengeable sequents $\Theta \succ A$ in π. For the basis case suppose that we have a challengeable sequent $\Theta \succ A$ the height above which is 0. This means that the stack ending in $\Theta \succ A$ cannot contain an E-segment. Furthermore, as $\Theta \succ A$ is challengeable it follows that $\Theta \neq [A]$. It is easy to see that this means that the stack ending in $\Theta \succ A$ must be of the following form:

$$p_i \succ p_i, \succ p_i \to p_i$$

so that $A = p_i \to p_i$ for some formula propositional variable p_i, and Θ is empty. The S-challenge of $\succ p_i \to p_i$ is the sequent $p_i \succ p_i$, which Prover can derive from itself using the elimination rules and structural rules 0 times. So we have the following winning strategy for Prover:

$$\begin{array}{c} | \\ S: \ (p_i \succ p_i) \\ | \\ P: \ (\{p_i \succ p_i\}) \end{array}$$

As the Skeptic cannot challenge an axiom, this strategy is a winning strategy for Prover.

For the induction hypothesis suppose that for all challengeable sequents $\Xi \succ B$ the height above which is $\leq n$ we have a winning strategy $D[\Xi \succ B]$. Suppose now that we have a challengeable sequent $\Theta \succ A$ the height above which is $n+1$. Then the Skeptic challenge for $\Theta \succ A$ will be $\Theta, \Theta' \succ p_i$, where p_i is the rightmost propositional variable in A. It follows that as π is in top heavy form that this is the minimum sequent in the stack ending in $\Theta \succ A$. Let $A_0 \succ A_0$ be the first sequent in the stack T ending in $\Theta \succ A$, and let C be the multiset of sequents $\Gamma_1^* \succ A_1^*, \ldots, \Gamma_n^* \succ A_n^*$ where each of the $\Gamma_i^* \succ A_i^*$ is a minor premise of an application of $(\rightarrow E)$ whose major premise is in T. Then it is clear that

$$[A_0 \succ A_0] \cup C \Rightarrow^*_{(E)} \Theta, \Theta' \succ p_i$$

Moreover, each of the challengeable formulas in C is the final formula of a track the height above which is n, and so by the induction hypothesis we have a winning strategy for all such sequents. This yields the following winning strategy for Prover, where $\Gamma_1' \succ A_1', \ldots, \Gamma_m' \succ A_m'$ ($m \leq n$) are all the challengeable sequents in C

$$\begin{array}{c} | \\ S: \ (\Theta, \Theta' \succ p_i) \\ | \\ P: \ ([A_0 \succ A_0] \cup C) \end{array}$$

$$D[\Gamma_1' \succ A_1'] \qquad \cdots \qquad D[\Gamma_m' \succ A_m']$$

Pasting together all these strategies, and adding $S : ([\Gamma \succ A])$ will yield a winning strategy for $\Gamma \succ C$. Note, also, that as we have generated this strategy from an S-derivation of $\Gamma \succ C$ that each branch of the winning strategy is an S-dialogue, and thus we have a winning S-strategy. □

As mentioned earlier, the above results make use of a more liberal conception of what constitutes a Prover move, where they are not *required* to make explicit use of the Skeptic's offer in order to derive the challenge. As it happens, though, the above proofs can also be use to show that we could have used the following, stricter, understanding of what a Prover move consists in:

- A Prover move S_{i+1} responding to a Skeptic challenge $\beta_i = \Gamma \succ p$ consists of them offering a multiset of sequents $\Gamma_j \succ A_j$ where:

– For some sentence $\alpha = A_1 \to (\ldots \to (A_m \to p)\ldots)$ we have $\alpha \succ \alpha \in S_{i+1}$, where $\alpha \in \Gamma$.
– The remaining sequents in S_{i+1} are of the form $\Gamma_j \succ A_j$.
– The multiset union of the Γ_js should be:
  * Identical to Γ in a linear dialogue
  * Be a sub-multiset of Γ in an affine dialogue.
  * Have the same underlying set as Γ in a relevant dialogue
  * Otherwise the underlying set of the multiset union of the Γ_js should be a subset of the underlying set of Γ.

Call a dialogue in which Prover makes use of only moves like that above a *strict dialogue*. Then we have already essentially shown the following.

Proposition A.12 ('Soundness & Completeness for Strict Dialogues'). *Let S* $\subseteq \{(W), (K)\}$. *Then,* $\vdash_S \Gamma \succ C$ *if and only if there is a winning strategy for Prover over the strict S-dialogue* (Γ, C).

Proof. Completeness is exactly as in Proposition A.3. For soundness note that by Definition A.5 that the sequent $A_0 \succ A_0$ appealed to in the proof of Proposition A.11 will be the major premise of an application of Modus Ponens, and so must be of the form $A_1 \to (\ldots \to (A_m \to p_i)\ldots)$ as desired. □

Strict Dialogues are useful for demonstrating that Prover cannot have a winning strategy for certain sequents.