In the Mood for **S4**: The Expressive Power of the Subjunctive Modal Language in Weak Background Logics*

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Abstract

Our concern here is with the extent to which the expressive equivalence of Wehmeier's Subjunctive Modal Language (SML) and the Actuality Modal Language (AML) is sensitive to the choice of background modal logic. In particular we will show that, when we are enriching quantified modal logics weaker than **S5**, AML is strictly expressively stronger than SML, this result following from general considerations regarding the relationship between operators and predicate markers. This would seem to complicate arguments given in favour of SML which rely upon its being expressively equivalent to AML.

1 Introduction

The expressive inadequacies of the standard modal language are well known, caused in part by a lack of any device for forcing the rigid evaluation of formulas at the actual world. The typical way of overcoming these expressive inadequacies, the AML approach, has been to enrich the language of modal logic with an actuality operator '$\mathbb{A}$', which forces formulas within its scope to be evaluated at the actual world. This is not the only way of enriching the expressive capacities of the standard modal language, an alternative way of doing so having been argued for by Kai Wehmeier in a series of papers beginning with [18]. This alternate approach, which following Wehmeier we will call the SML approach,[1] enriches the expressive power of our modal language with a class of marked predicates, with the unmarked predicates evaluated rigidly at the actual world, and the marked predicates evaluated at the world of evaluation. Despite the initial thought one might have that, due to the additional

*This is a preprint. The final paper will be published in *Studia Logica* [HERE]. Readers should be aware that a number of notational changes happened between this version and the final version!

[1]On p.615 of [18] SML is taken to be an abbreviation for 'Subjunctive Modal Logic' (and similarly for AML). Throughout we will take this to be an abbreviation for the Subjunctive Modal Language, so as to be able to talk about variation in the background assumptions concerning the modal operators (i.e. variation in the background modal logic).

freedom of occurrence which an operator enjoys, the AML language might be expressively stronger than the SML one, Wehmeier shows therein that, in a very precise sense, these two languages are expressively equivalent when our background modal logic is **S5**.

Given this we are faced with a methodological question of how to choose between these two competing frameworks. There appear to be two main issues at stake in the dispute between AML and SML theorists—the significance of subjunctive closure, and the distinction between operators and predicate markers. Intuitively speaking, a formula is *subjunctively closed* iff every occurrence of a predicate in that formula is either within the scope of a modal operator, or is an 'indicative formula'—e.g. a formula either under the scope of an actuality operator in the AML case, or an unmarked predicate in the SML case. Arguments in favour of taking only subjunctively closed formulas as meaningful (analogous to similar arguments in favour of treating only closed sentences as meaningful) are given on p.616 of [18]. On this front, Wehmeier has argued that we should prefer SML over AML, as the actuality language creates spurious distinctions in the logical form of standard subject-predicate sentences—rendering sentences like "John is clever" ambiguously as either Cj or $\mathbb{A}Cj$. On the face of it this seems to be a rather telling indictment of the AML approach, but of course this issue can be resolved by the AML theorist by placing greater emphasis on the subjunctively closed fragment of their language, regarding formulas like Cj as having the same status as open formulas in predicate logic (cf. [4]). We will largely be unconcerned with this issue here, though.

Our main concern in this paper will be with the second of the two issues raised above: whether we should prefer interpreted languages with predicate markers over expressively equivalent ones with operators. After having just compared the AML and SML approaches to dealing with the expressive inadequacies of quantified modal logic, Lloyd Humberstone makes the following remark on this issue (crediting the sentiment expressed to Timothy Smiley).

> As a general rule, it is not a good idea to argue against a more comprehensive language—such as one in which ... ['$\mathbb{A}$'] can operate on an arbitrary formula and not just (in effect) on an atomic formula—and [for] a less comprehensive one, on the basis of an observation that everything that can be said in the richer language has an equivalent in the poorer, since if we work only with the poorer language, we can no longer formulate the observation in question. [10, p.46f]

In what follows we will make explicit the objection being made above for why one should not prefer such a less comprehensive language over a more comprehensive one. What we will show is that the aforementioned inability to express what is under dispute puts the predicate marker language at the mercies of the behaviour of parts of the language which are not under dispute. In particular we will focus on the case where we weaken the background modal logic, or enrich the object language in various ways. What we will argue here is that the expressive equivalence of the SML and AML frameworks is only an artefact of comparing the two different linguistic frameworks in a particular logical setting—namely first-order **S5** with the standard modal operators $\Box$ and $\Diamond$. The majority of work on actuality and subjunctivity has focused on this case,[2] but it turns out that variations on this logical setting make a striking difference to the equivalence of the AML and SML approaches. Moreover, when we change the framework it becomes unclear what precisely we are to take the SML approach to be against weaker background modal logics.

2 Preliminaries

Intuitively speaking, two languages L_1 and L_2 are *expressively equivalent* whenever they are able to discriminate between the same models. We will find it convenient to cash that out, here, in the following strong sense.

Definition 2.1. Two languages L_1 and L_2 are *expressively equivalent* iff we have the following:

$$\forall \varphi \in L_1\ \exists \psi \in L_2 (\forall \mathcal{M} : \mathcal{M} \models_{L_1} \varphi \iff \mathcal{M} \models_{L_2} \psi).$$
$$\forall \psi \in L_2\ \exists \varphi \in L_1 (\forall \mathcal{M} : \mathcal{M} \models_{L_1} \varphi \iff \mathcal{M} \models_{L_2} \psi).$$

Where $\mathcal{M} \models_{L_i} \varphi$ means that φ is true in the model $\mathcal{M}$ in the L_i-sense.

It is easy to see that this is the notion of expressive equivalence which is under discussion in [18, p.619] when it is noted that "from a purely logical point of view, ... [AML and SML] are expressively equivalent", the above definition having been used in the preceding paragraphs to show this.

Here our interest is in comparing the expressive power of SML and AML as different linguistic frameworks for correcting the expressive inadequacy of standard quantified modal logic. As a result we want to be able to separate issues concerning how the indicative mood is treated from issues concerning the logical behaviour of the modal operators. Consequently, we will draw a distinction between *linguistic frameworks* and *interpreted languages* as used in the above definition, with linguistic frameworks leaving some element of the interpretation of parts of the language under-defined. In our particular case what both SML and AML leave undefined is the logic of the modal operators—the background modal logic to which the actuality operator or indicative and subjunctive predicates are being added to. As a result all of our comparisons will be between SML and AML relative to some filling in of the background modal logic S. Formally speaking say that L_1 *is expressively equivalent to* L_2 *over* S whenever L_1 is expressively equivalent to L_2 whenever we restrict the class of models to those which satisfy S. This kind of relativity to a background modal logic is easily masked when the background modal logic is taken to be **S5**, as in

[2]Notable exceptions to this are [7], [15] (which discusses indexed actuality operators in logics weaker than **S5**) and [8] (which discusses propositional logics with the actuality operator). [20] is a further exception here, considering how the predicate marker and operator approaches are able to deal with cross-world predication, and finding in favour of the predicate marker approach. In deciding between AML and SML we will need to assess how the two frameworks can deal with cross-world predication, but this is something we will leave aside here.

this case rather than restrict ourselves to models with reflexive, transitive, and symmetric accessibility relations, we can instead change the truth-in-a-model clause for $\Diamond$-formulas in both languages so that $\Diamond\varphi$ is true at a world w just in case φ is true at some world w' in the model. Once this kind of modification is made, we end up with a straightforward comparison of expressive power between two different interpreted languages.

Another subtlety which we have to be aware of when considering comparisons of expressive power in a modal setting is what we take truth in a model to consist in for our languages L_i. Here we will, following [18, p.619], take truth in a model for languages in the AML framework to be truth at the actual world in the model. The core result of §5, that SML is expressively weaker than AML when our background modal logic is weaker than **S5**, follows not only if we take truth in a model for languages in the SML framework to be truth at some world in a model (as suggested in [18, p.618], but also if we let truth in a model be truth at the actual world.

In moving from **S5** to weaker logics like **S4** we also have to take care over how we are altering our models to allow for the evaluation of the actuality operator and indicative predicates. That is, in the terminology of [8], we have to take care regarding what we take the *actuality extension* of a given logic to be. This is important because many of the requirements one might impose upon how we are to extend our models to allow for the actuality operator are equivalent in **S5**, but not in weaker logics.

For example in [16] we are presented with a number of different conditions which we can place upon which worlds are accessible to and accessible from the actual world @ in a model. For example we could place either of the following two constraints upon our models.

$$\text{For every world } w,\ R@w. \tag{1}$$

$$\text{For every world } w,\ Rw@. \tag{2}$$

In the case where our background modal logic extends **KB** conditions (1) and (2) are equivalent, but are distinct in logics such as **S4** which do not extend **KB**. The reason we mention this is just to point out the extra degrees of freedom which arise when we are considering how to enrich logics like **S4** by the addition of the actuality operator. In [8] a general recipe is given for going from a given modal logic S characterised by a class of frames $\mathcal{C}$ to the class of frames $\mathcal{C}_{\mathbb{A}}$ which determines its actuality extension $\mathsf{S}_{\mathbb{A}}(\mathcal{C})$ w.r.t the class of frames $\mathcal{C}$. How this works is most easily described in the propositional case, but it will be obvious how this can be extended to frames for first-order modal logics. Given a class of frames $\mathcal{C}$ define $\mathcal{C}_{\mathbb{A}}$ to be $\{\langle W, w, R\rangle | \langle W, R\rangle \in \mathcal{C}, w \in W\}$. That is to say, the actuality extension of a class of frames results from selecting each point in each frame in turn as the distinguished point. Of course, in the first-order case we are not guaranteed that when a logic S is characterised by multiple different classes of frames that their actuality extensions will coincide. In this case, where S is a logic determined by at least one class of frames, let us say that the *actuality* *extension of a logic* S is the logic determined by the actuality extension of the class of all frames for S. We bring this up mostly to make clear that the frames which we will be discussing below in §5, which satisfy condition (1) above, are amongst the actuality extension of first-order **S4**.

3 Introducing AML and SML

Letting P be a set of predicate symbols, the language of L has as primitive symbols the elements of P, the equality symbol $=$, the boolean connectives $\{\neg, \wedge\}$, denumerably many individual variables $x, y, x_1, x_2, \ldots$, and the existential quantifier $\exists$. The language QL_A additionally has the modal operators $\Diamond$ and and $\mathbb{A}$, as well as an 'actuality quantifier' $\exists^{@}$. The language QL_S has the modal operator $\Diamond$, an 'actuality quantifier' $\exists^{@}$ and, for each predicate $P \in \mathsf{P}$ the subjunctive predicate P^s. Here QL_A represents the formal language of the AML framework, and QL_S that of the SML framework. The language we are calling QL_S here is a notational variant of Wehmeier's, who, rather than having an explicitly marked actuality quantifier (or indicative quantifier in Wehmeier's terms) and an unmarked subjunctive quantifier, instead has an explicitly marked subjunctive quantifier $\exists^s$ and treats the unmarked quantifier indicatively. We use this language in order to ease the cognitive load on the reader.[3]

The formulas of QL_S and QL_A are defined inductively as follows.

QL_S formulas

- If P is an n-ary predicate from P and $x_1, \ldots, x_n$ are individual variables then $Px_1 \ldots x_n$ is a QL_S formula.
- If P is an n-ary predicate from P and $x_1, \ldots, x_n$ are individual variables then $P^s x_1 \ldots x_n$ is a QL_S formula.
- If x and y are individual variables then $x = y$ is a QL_S formula.
- φ and ψ are formulas of QL_S, then so are $\neg\varphi$ and $\varphi \wedge \psi$.
- If φ is a formula and x an individual variable, then $\exists x\varphi$ and $\exists^{@} x\varphi$ are QL_S formulas.
- If φ is a QL_S formula then so is $\Diamond\varphi$.

[3] Wehmeier's reasons for making this particular notational choice are clearly articulated in [19, p.199] where he says the following: "An obvious constraint on any language of modal predicate logic is that its non-modal part should simply be the language of ordinary predicate logic. Therefore, indicative predicates are to be expressed by the ordinary predicate symbols of non-modal predicate logic (which, after all, formalises ordinary, indicative discourse), and it is the subjunctive for which we need to introduce a new notation." That this is an *obvious* constraint is not clear to this author.

QL_A **formulas**

- If P is an n-ary predicate from P and $x_1, \ldots, x_n$ are individual variables then $Px_1 \ldots x_n$ is a QL_A formula.
- If x and y are individual variables then $x = y$ is a QL_A formula.
- φ and ψ are formulas of QL_A, then so are $\neg\varphi$ and $\varphi \wedge \psi$.
- If φ is a formula and x an individual variable, then $\exists x\varphi$ and $\exists^{@} x\varphi$ are QL_A formulas.
- If φ is a QL_A formula then so is $\Diamond\varphi$ and $\mathbb{A}\varphi$.

A sentence of a given language is a closed formula from that language, where closed formulas are understood standardly as formulas in which every individual variable occurring in the formula is bound by a quantifier. A formula in QL_S is *subjunctively closed* if every occurrence of a subjunctive predicate symbol and subjunctive quantifier '$\exists$' is in the scope of a modal operator.

Our semantic structures will be expansions of Kripke frames for quantified **S4** where we select a distinguished world $@ \in W$. More precisely, a model will be a tuple $\mathcal{M} = \langle W, @, R, (D_w)_{w\in W}, (P_w)^{P\in\mathsf{P}}_{w\in W}\rangle$, where R is a reflexive, transitive relation on W, $(D_w)_{w\in W}$ is a W-indexed family of sets at least one of which is non-empty (the union of which is denoted by D) and a doubly indexed family $(P_w)^{P\in\mathsf{P}}_{w\in W}$ of relations, where for each n-ary $P \in \mathsf{P}$ and arbitrary $w \in W$, P_w is a subset of D^n.[4] A *frame* will, as is standard, be taken to be a structure $\mathcal{F} = \langle W, @, R, (D_w)_{w\in W}\rangle$.

A variable assignment σ is a mapping from the set of individual variables into D, and $\sigma\{x := o\}$ is the function which is like σ except that it maps the variable x to the object o.

Truth of a formula φ at a point w in a model $\mathcal{M}$ on a variable assignment σ for formulas in QL_A ("$\mathcal{M}, \sigma \models_w \varphi$") will be defined inductively as follows.

$$
\begin{aligned}
\mathcal{M}, \sigma \models_w Px_1 \ldots x_n &\iff \langle\sigma(x_1), \ldots, \sigma(x_n)\rangle \in P_w.\\
\mathcal{M}, \sigma \models_w x_i = x_j &\iff \sigma(x_i) = \sigma(x_j).\\
\mathcal{M}, \sigma \models_w \varphi \wedge \psi &\iff \mathcal{M}, \sigma \models_w \varphi \text{ and } \mathcal{M}, \sigma \models_w \psi.\\
\mathcal{M}, \sigma \models_w \neg\varphi &\iff \mathcal{M}, \sigma \not\models_w \varphi.\\
\mathcal{M}, \sigma \models_w \Diamond\varphi &\iff \text{for some } v \in W\text{: } Rwv \text{ and } \mathcal{M}, \sigma \models_v \varphi.\\
\mathcal{M}, \sigma \models_w \mathbb{A}\varphi &\iff \mathcal{M}, \sigma \models_@ \varphi.\\
\mathcal{M}, \sigma \models_w \exists x\varphi &\iff \text{for some } o \in D_w : \mathcal{M}, \sigma\{x := o\} \models_w \varphi.\\
\mathcal{M}, \sigma \models_w \exists^{@} x\varphi &\iff \text{for some } o \in D_@ : \mathcal{M}, \sigma\{x := o\} \models_w \varphi.
\end{aligned}
$$

In altering SML from the setting where the background modal logic is **S5** in order to accommodate changes in the background modal logic we are faced

[4]Note that officially we place no restrictions on the sets D_w, although as it happens throughout our models are such that if Rwv then $D_w \subseteq D_v$, and this fact will be used essentially in our result.

with a choice in how we are to evaluate formulas of the form $\Diamond\varphi$, as there are two different clauses which are compatible with the behaviour of $\Diamond$ in the SML framework when our background modal logic is **S5**. Either we can say that (i) $\Diamond\varphi$ is true at a world w iff there is a world accessible *from* w at which φ is true, or (ii) that $\Diamond\varphi$ is true at a world w iff there is a world accessible *from* @ at which φ is true. This creates a dilemma for the SML theorist: they must either treat their modal operators 'indicatively' (and so follow option (ii)) above, or treat them 'subjunctively' (and follow option (i)) above. Here we will take the obvious way of lifting the language to be the standard one (corresponding to option (i) above), and thus begin in the following section by discussing how this option fares, examining the indicative treatment in §8.[5]

As a result of this dilemma we will need to define two different notions of truth in a model for formulas from QL_S, corresponding to two different interpreted languages, which differ over their evaluation of $\Diamond$-formulas: $\Vdash$ for the subjunctive interpretation, and $\Vdash^i$ for the indicative interpretation (which we will define in §8). Which of these two definitions of truth in a model the SML theorist should favour is not clear, as we will see both of them run into various difficulties. Our dialectical purpose here is to discuss the relative costs of these two different ways the SML theorist has of handling weaker background modal logics.

Truth of a formula φ at a point w in a model $\mathcal{M}$ relative to a variable assignment σ for formulas in QL_S ("$\mathcal{M}, \sigma \Vdash_w \varphi$") will be defined inductively as follows.

$$\begin{array}{rcl}
\mathcal{M}, \sigma \Vdash_w Px_1 \ldots x_n & \iff & \langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in P_@. \\
\mathcal{M}, \sigma \Vdash_w P^s x_1 \ldots x_n & \iff & \langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in P_w. \\
\mathcal{M}, \sigma \Vdash_w x_i = x_j & \iff & \sigma(x_i) = \sigma(x_j). \\
\mathcal{M}, \sigma \Vdash_w \varphi \wedge \psi & \iff & \mathcal{M}, \sigma \Vdash_w \varphi \text{ and } \mathcal{M}, \sigma \Vdash_w \psi. \\
\mathcal{M}, \sigma \Vdash_w \neg\varphi & \iff & \mathcal{M}, \sigma \nVdash_w \varphi. \\
\mathcal{M}, \sigma \Vdash_w \Diamond\varphi & \iff & \text{for some } v \in W\text{: } Rwv \text{ and } \mathcal{M}, \sigma \Vdash_v \varphi. \\
\mathcal{M}, \sigma \Vdash_w \exists x\varphi & \iff & \text{for some } o \in D_w : \mathcal{M}, \sigma\{x := o\} \Vdash_w \varphi. \\
\mathcal{M}, \sigma \Vdash_w \exists^@ x\varphi & \iff & \text{for some } o \in D_@ : \mathcal{M}, \sigma\{x := o\} \Vdash_w \varphi.
\end{array}$$

4 Subjunctive-SML Morphisms

We will begin by considering how the subjunctive interpretation of SML (henceforth SML-s) fares. In order to address the question of whether SML-s and AML are expressively equivalent over weaker background logics we will first need to establish some preliminary results concerning morphisms between predicate Kripke frames which preserve SML-formulas. We will henceforth, for the sake of simplicity, restrict ourselves to frames with cumulative domains—frames

[5]It is worth noting that Wehmeier (p.c.) does not share this intuition regarding the plausibility concerning how to treat the modal operators in weaker settings.

where if Ruv then $D_u \subseteq D_v$. The following definition is an adaption of that given in [6, p.219f].

Definition 4.1. Suppose that $\mathcal{F} = \langle W, R, @, (D_w)_{w\in W}\rangle$ and $\mathcal{F}' = \langle W', R', @', (D'_w)_{w\in W'}\rangle$ are predicate Kripke frames. Then an *SML-frame-morphism* from $\mathcal{M}$ to $\mathcal{M}'$ is a pair $\mathbf{f} = \langle f, g\rangle$ s.t.

1. f is a surjection such that $f(@) = @'$.
2. $\forall u, v \in W$ if Ruv then $R'f(u)f(v)$.
3. $\forall u \in W\ \forall v' \in W'$ if $R'f(u)v'$ then $\exists v(Ruv\ \&\ f(v) = v')$.
4. g is a W indexed set of functions g_u for $u \in W$.
5. For all $u \in W$,$g_u : (D_u \cup D_@) \to (D'_{f(u)} \cup D'_{f(@)})$ are bijections s.t. $g_u(D_u) = D'_{f(u)}$ and $g_u(D_@) = D'_{f(@)}$.
6. For all $u, v \in W$ if Ruv then for all $a \in D_u$ $g_u(a) = g_v(a)$.

The first three conditions here suffice to make f a p-morphism between the underlying propositional frames of $\mathcal{M}$ and $\mathcal{M}'$ (with the added condition that @ is mapped to @′). The final three conditions are the more interesting ones, here. These conditions are required in order to ensure that frames which are related by a SML-morphism validate the same quantificational formulas.

Definition 4.2. An *SML-model-morphism* is a SML-frame-morphism $\mathbf{f} = \langle f, g\rangle$ from predicate Kripke models $\mathcal{M} = \langle W, R, @, (D_w)_{w\in W}, (P_w)^{P\in\mathsf{P}}_{w\in W}\rangle$ and $\mathcal{M}' = \langle W', R', @', (D'_w)_{w\in W'}, (P'_w)^{P\in\mathsf{P}}_{w\in W'}\rangle$, an *SML-model-morphism* from $\mathcal{M}$ to $\mathcal{M}'$ which also satisfies the following condition for all n-ary predicates $F \in \mathsf{P}$ and all $w \in W$:

$$\langle a_1, \ldots, a_n\rangle \in F_w \iff \langle g_w(a_1), \ldots, g_w(a_n)\rangle \in F'_{f(w)}$$

Definition 4.3. Given a predicate Kripke frame $\mathcal{M} = \langle W, R, @, D\rangle$, say that a variable assignment σ is a *u-assignment for φ* iff for all variables x_i which appear free in φ we have that $\sigma(x_i) \in (D_u \cup D_@)$.

Theorem 4.4. *If $\mathbf{f}$ is a SML-model-morphism from the predicate Kripke model $\mathcal{M} = \langle W, R, @, (D_w)_{w\in W}, (P_w)^{P\in\mathsf{P}}_{w\in W}\rangle$ to the predicate Kripke model $\mathcal{M}' = \langle W', R', @', (D'_w)_{w\in W'}, (P'_w)^{P\in\mathsf{P}}_{w\in W'}\rangle$ then, for all $u \in W$ and all σ which are u-assignments for φ:*

$$\mathcal{M} \Vdash_u \varphi[\sigma] \text{ if and only if } \mathcal{M}' \Vdash_{f(u)} \varphi[g_u(\sigma)].$$

Proof. By induction on the complexity of φ.

Basis Case: There are three cases in the basis case: that where φ is $Fx_1 \ldots x_n$, where φ is $F^s x_1 \ldots x_n$, and that where φ is $x = y$. The cases where φ is an indicative or subjunctive predication are exactly similar and follow from

f being a morphism between the models. That is, $\mathcal{M} \Vdash_u Fx_1 \ldots x_n[\sigma]$ iff $\langle \sigma(x_1), \ldots \sigma(x_n) \rangle \in F_@$ iff $\langle g_@(\sigma(x_1)), \ldots, g_@(\sigma(x_n)) \rangle \in F'_{f(@)}$ iff $\mathcal{M}' \Vdash_{f(u)} Fx_1 \ldots x_n[g_@(\sigma)]$ iff (as $g_@$ and g_u agree on $D_@$) $\mathcal{M}' \Vdash_{f(u)} Fx_1 \ldots x_n[g_u(\sigma)]$. For the case where φ is $x = y$ suppose that $\mathcal{M} \Vdash_u x = y[\sigma]$. This is the case iff $\sigma(x) = \sigma(y)$, which as g_u is a bijection from $D_u \cup D_@$ to $D'_{f(u)} \cup D'_{f(@)}$ will be the case iff $g_u(\sigma(x)) = g_u(\sigma(y))$ iff $\mathcal{M}' \Vdash_{f(u)} x = y[g_u(\sigma)]$.

Inductive Step: The interesting cases in the inductive step are those where for some formula ψ, φ is either $\exists x\psi$, or $\exists^@ x\psi$, or $\Diamond\psi$. We treat each case in turn.

$\exists x\psi$ **Case:** Suppose that $\mathcal{M} \Vdash_u \exists x\psi[\sigma]$. This is the case iff there is an $a \in D_u$ and a σ' s.t. $\sigma' \sim_x \sigma$ and $\sigma'(x) = a$ and $\mathcal{M} \Vdash_u \psi[\sigma']$. As σ' is a u-assignment for ψ it follows by the induction hypothesis that $\mathcal{M}' \Vdash_{f(u)} \psi[g_u(\sigma')]$. By condition (5) of Definition 4.1 g_u is a bijection from D_u to $D'_{f(u)}$, thus as a $(= g_u(\sigma')(x))$ is in D_u it follows that $g_u(\sigma')$ is an x-variant of $g_u(\sigma)$ s.t. $g_u(\sigma')(x) \in D'_{f(u)}$. Consequently, $\mathcal{M}' \Vdash_{f(u)} \exists x\psi[g_u(\sigma)]$ as desired.

$\exists^@ x\psi$ **Case:** Suppose that $\mathcal{M} \Vdash_u \exists^@ x\psi[\sigma]$. This is the case iff there is an $a \in D_@$ and a σ' s.t. $\sigma' \sim_x \sigma$ and $\sigma'(x) = a$ and $\mathcal{M} \Vdash_u \psi[\sigma']$. As σ' is a u-assignment for ψ it follows by the induction hypothesis that $\mathcal{M}' \Vdash_{f(u)} \psi[g_u(\sigma')]$. By condition (5) of Definition 4.1 g_u is a bijection from $D_@$ to $D'_{f(@)}$, thus as a $(= g_u(\sigma')(x))$ is in $D_@$ it follows that $g_u(\sigma')$ is an x-variant of $g_u(\sigma)$ s.t. $g_u(\sigma')(x) \in D'_{f(@)}$. Consequently, $\mathcal{M}' \Vdash_{f(u)} \exists^@ x\psi[g_u(\sigma)]$ as desired.

$\Diamond\psi$ **Case:** Follows from the fact that f is a p-morphism.

□

5 Two Equivalent SML-models

We will now use the material from the pervious section to show that SML-s and AML are not expressive equivalent over weaker background logics. Consider the following two models, where $W = \{@, w_0, w_1\}$ and R is the least reflexive, transitive relation on W such that $R@w_0$ and $R@w_1$.

- $\mathcal{M} = \langle W, R, @, D^M, F^M \rangle$

  $D^M = \{\langle @, \{o\}\rangle, \langle w_0, \{a, o\}\rangle, \langle w_1, \{b, o\}\rangle\}$.

  $F^M = \{\langle @, \emptyset\rangle, \langle w_0, \{a\}\rangle, \langle w_1, \{b\}\rangle\}$.

- $\mathcal{N} = \langle W, R, @, D^N, F^N \rangle$

  $D^N = \{\langle @, \{o\}\rangle, \langle w_0, \{a, o\}\rangle, \langle w_1, \{a, o\}\rangle\}$.

  $F^N = \{\langle @, \emptyset\rangle, \langle w_0, \{a\}\rangle, \langle w_1, \{a\}\rangle\}$.

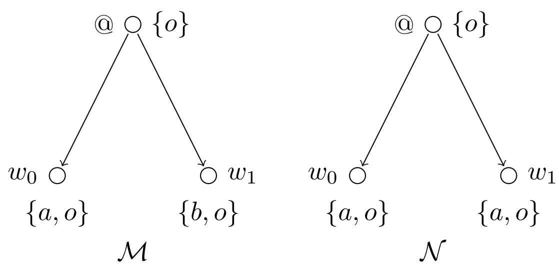

Figure 1: The Models $\mathcal{M}$ and $\mathcal{N}$, the open circles indicating reflexive points.

It is easy to see that the following QL_A-sentence is true at @ in $\mathcal{M}$, and false at @ in $\mathcal{N}$.

$$(\star) \quad \Diamond[\exists x(Fx \wedge \mathbb{A}\Diamond(\exists y(Fy \wedge x \neq y)))].$$

Kai Wehmeier has pointed out to the author in conversation that this formula bears more structurally similarity to Kamp-style tense logical sentences like "A child was born who will become ruler of the world", traditionally used to show the expressive inadequacy of quantified tense logic without the 'Now' operator [11, p.231], than to the examples of the form "It is possible that every red thing should be shiny" considered in e.g. [9], and [18]. To see this write the past and future tense operators as $\Diamond^-$ and $\Diamond^+$ respectively (rather than the Priorian P and F) and consider the Kamp-sentence's logical form:

$$\Diamond^-[\exists x(\mathit{Born}(x) \wedge \mathsf{N}\Diamond^+(\mathit{Ruler\text{-}of\text{-}the\text{-}world}(x)))].$$

This suggests that there are two different kinds of expressive weakness which get conflated when our background modal logic is **S5**. On the one hand we have expressivity issues which concern *rigid predicate evaluation*. These are usually motivated by wanting to treat natural language sentences like "It is possible that every red thing should be shiny", where we need a device like the actuality operator or predicate-markers in order to evaluate certain predicates (in this case 'red') at the actual world—taking them outside the semantic scope of modal operators governing them. On the other hand we have expressivity issues which concern *resetting the world of evaluation*. This is the kind of expressive weakness brought out by the Kamp-sentence above, where we need to evaluate a certain formula from the present time—taking an arbitrary, as opposed to merely atomic, formula outside the semantic scope of the modal/temporal operators governing it. For example, in the Kamp-sentence (reverting to Priorian notation) we need to take '$\mathsf{F}(\mathit{Ruler\text{-}of\text{-}the\text{-}world}(x))$' outside the semantic scope of the P. It is worth noting that in all the apparent ways of treating this kind of expressive weakness also allow for rigid predicate evaluation as a special case.

Focusing on background logical frameworks like **S5** where, in a certain sense, all worlds are equal, the second kind of potential expressive weakness has gone largely unnoticed in the modal case. In the tense logical case the difference between the two kinds of expressive weakness would appear to have gone unnoticed for a different reason. As the Kamp-sentence above shows the immediate need in tense logic is to be able to be able to reset the evaluation of a formula back to the present time. Thus, in the tense logical case, we end up having to treat this other kind of expressive weakness, of which rigid predicate evaluation is a special case.

Now, consider the following W-indexed set of functions $(g_u)_{u \in W}$.

- $g_@$ is the identity map.
- g_{w_0} is the identity map.
- $g_{w_1}(o) = o$; $g_{w_1}(b) = a$

It is easy to see the following.

Lemma 5.1. *The pair $\langle id, g \rangle$ is an SML-model-morphism from $\mathcal{M}$ to $\mathcal{N}$, where id is the identity function.*

Proposition 5.2. *The models $\mathcal{M}$ and $\mathcal{N}$ verify the same SML formulas.*

Proof. Follows from the above Lemma and Theorem 4.4. □

Corollary 5.3. *For all sentences $\varphi \in QL_S$ we have that:*

$$\mathcal{M} \Vdash_@ \varphi \textit{ if and only if } \mathcal{N} \Vdash_@ \varphi.$$

What this result show is that there is no sentence (i.e. closed formula) which discriminates between the models $\mathcal{M}$ and $\mathcal{N}$, no matter whether truth in a model is truth at some world in the model, truth at all worlds in the model, or truth at the actual world in the model (these all just being various special cases of Proposition 5.2, just like Corollary 5.3 above).

There are two important points which should be made here concerning the SML-s framework in weaker background modal logics. The first is that in such a setting formulas which are subjunctively closed are no longer true at all worlds in a model if they are true at at least one world. This means that the notion of truth in a model for the SML language is no longer so clearly motivated by the requirement that the only 'first class' logical citizens are subjunctively closed sentences, as is claimed in [18, p.618].[6] This is a further illustration of

[6] It is worth noting, though, that when we are construing the modal operators in the SML-i sense discussed in §8 that subjunctively closed formulas are true throughout a model whenever they are true at a point in the model. In a language with mixed indicative and subjunctive modal operators the SML theorist would have to further amend the definition of subjunctive closure to require that subjunctive modal operators only occur within the scope of indicative ones in order to keep the modal invariance of subjunctively closed formulas. At this point the present author feels we are starting to drift further away from the intuitive motivation for the notion given in [18].

the phenomenon which we discussed in §2 in which notions which are equivalent in **S5** come apart in weaker modal logics—as we now can draw a distinction between three different notions of truth in a model for the SML language: truth at a world, truth throughout the model, truth at the actual world. The above results are invariant as regards the background notion of truth in a model.[7]

On the basis of this we have the following core result.

Theorem 5.4. *QL_A and the SML-s version of QL_S are not expressively equivalent over* **S4**.

Proof. The result follows from Corollary 5.3 and the fact that $(\star)$ is true in $\mathcal{M}$ but not in $\mathcal{N}$. $\square$

That this indicates a particular expressive deficit on the part of QL_S can be seen in light of the following result.

Theorem 5.5. *QL_A is at least as expressive as the SML-s version of QL_S over all modal logics* S.

Proof. To see this consider the following translation τ, which is homonymous on the boolean connectives.

$$\tau(Fx_1 \dots x_n) = \mathbb{A}Fx_1 \dots x_n;\ \tau(F^s x_1 \dots x_n) = Fx_1 \dots x_n;$$

$$\tau(\exists x\varphi) = \exists x\tau(\varphi);\ \tau(\exists^{@} x\varphi) = \exists^{@} x\tau(\varphi)$$

It follows by an easy induction on the complexity of $\varphi \in QL_S$ that for all models $\mathcal{M}$, worlds w and assignments σ that

$$\mathcal{M} \Vdash_w \varphi[\sigma] \text{ if and only if } \mathcal{M} \models_w \tau(\varphi)[\sigma],$$

which is all that is required to show that QL_A is at least as expressive as QL_S. $\square$

Thus, QL_A is strictly more expressive that the SML-s version of QL_S over **S4**. Our reason for singling out **S4** in the above results is mostly one of philosophical plausibility and familiarity. As it happens the above result can be strengthened in a number of ways. For example, it also follows directly from the above results that QL_A and QL_S are not expressively equivalent if the background modal logic is $\mathbf{S4Alt}_3$—where we also require that for all w, $|R(w)| \leq 3$, where $R(w) = \{v | Rwv\}$. Similarly, we can let out background modal logic be **S4M**, or indeed any modal logic S for which the underlying frame of $\mathcal{M}$ (and thus $\mathcal{N}$) is a frame for S.

Furthermore, we can also extend our background modal logic by also adding principles which mix modal operators and the quantifiers. As D^M and D^N are both monotonic in R (in the sense that if Rwv then $D_w \subseteq D_v$), the above result

[7] Note that, unlike the cases considered in [5], this holds even if we allow sentences which are not subjunctively closed to be first-class logical citizens.

holds even if we extend the background modal logic so that the converse Barcan formula (CBF) is valid.

$$(CBF): \quad \exists x \Diamond Fx \rightarrow \Diamond \exists x Fx.^{8}$$

This suggests one way in which the proponent of SML can resist the above counterexample: to demand that all models have constant domains. If we are dealing with models with constant domains we will be able to represent ($\star$) in SML, because we can just move the quantifiers outside the scope of the $\Diamond$'s, allowing us to represent ($\star$) as:

$$(\exists^{@} x)(\exists^{@} y)[\Diamond Fx \wedge \Diamond Fy \wedge x \neq y].$$

Of course, this solution commits us to everything which exists doing so necessarily. Given that most people still tend to regard contingent existence, and hence variable-domain semantics, as more philosophically plausible, this fix comes at a cost in plausibility.[9]

5.1 What Fragment of AML has SML Equivalents

Having just shown that AML and SML-s are not equivalent over these weaker background modal logics, the question arises as to whether there is some *fragment* of AML which can be expressed in the language of subjunctive SML. To make this question clearer, say that a formula $\varphi \in LQ_A$ *has an SML-s-equivalent* iff there is a formula $\psi \in LQ_S$ s.t.

$$\forall \mathcal{M} : \mathcal{M} \models \varphi \iff \mathcal{M} \Vdash \psi.$$

The notion of having an L-equivalent is a weakening of the notion of expressive equivalence, two language L and L' being expressively equivalent whenever every L formula has an L'-equivalent and vise versa. What we want to know, then, is what subset of AML formulas have SML-s-equivalents.

Say that a formula $\varphi \in QL_A$ is in the *local-scope* fragment whenever every occurrence of the actuality operator in φ has scope only over primitive predications.

Proposition 5.6. *Suppose that $\varphi \in QL_A$ is a formula which has an SML-s-equivalent. Then φ is equivalent to a formula in the local-scope fragment of AML.*

Proof. Suppose that there is a formula $\psi \in LQ_S$ s.t.

$$\forall \mathcal{M} : \mathcal{M} \models \varphi \iff \mathcal{M} \Vdash \psi.$$

[8] This is not a subjunctively closed formula, of course. It is an interesting question what kinds of modal definability results are available in SML in terms of subjunctively closed formulas only. In this case we could make do with the formulas $\exists^{@} x \Diamond Fx \rightarrow \Diamond \exists x Fx$ and $\Box(\exists x \Diamond Fx \rightarrow \Diamond \exists x Fx)$ if we let our background modal logic be an extension of **K4**.

[9] Prime examples of dissenters from what we have painted as the philosophical orthodoxy are [13] and [22], who defend constant domain quantified modal logic.

As noted in Theorem 5.5 we know that:

$$\forall \mathcal{M} : \mathcal{M} \Vdash \psi \iff \mathcal{M} \models \tau(\psi).$$

So it follows, then that,

$$\forall \mathcal{M} : \mathcal{M} \models \varphi \iff \mathcal{M} \models \tau(\psi).$$

But $\tau(\psi)$ can readily be seen to be in the local-scope fragment of AML. $\square$

This result makes clear that the expressive deficit of SML-s entirely has to do with its inability, in these weaker fragments, to deal with genuine world travelling.

6 Monocosmic Operators and Predicate Markers

What is going on here is that in logics weaker than **S5**, we are not able to move modal operators out of the scope of the actuality operator. In describing what is going on it is worth introducing the idea of an operator O being *monocosmic* to use the term introduced in [10].[10] Say that an operator O is monocosmic whenever the following equivalences are all satisfied (relative to our semantic apparatus) for all n-ary boolean connectives #:

$$O\#(\varphi_1, \ldots, \varphi_n) \equiv \#(O\varphi_1, \ldots, O\varphi_n). \tag{3}$$

$$OO\varphi \equiv O\varphi. \tag{4}$$

$$O\Box\varphi \equiv \Box\varphi. \tag{5}$$

$$O\exists x\varphi \equiv \exists^O xO\varphi. \tag{6}$$

Here $\exists^O x\varphi$ is some kind of quantifier in the object language (most commonly in our case $\exists$ or $\exists^@$). The reason for being interested in an operator being monocosmic is that such operators can be replaced by a new sort of O-predicates, the above equivalences allowing all instances of O to be 'pushed inwards' to only have scope over primitive predications, at which point $OFt_1 \ldots t_n$ can be replaced with $F^O t_1 \ldots t_n$. The failure of the expressive equivalence of AML and SML arises here due to the fact that in **S4** the actuality operator cannot be pushed inwards over the box operator. As a result AML is expressively richer than SML in this setting as, conceiving of our models as trees, we are able to use the actuality operator to go back to the actual world and evaluate modal formulas down different branches of our model, while all we are able to do in SML is evaluate non-modal formulas at the actual world in the course of our world-travelling (to use the evocative metaphor of [17]).

The importance of this result to the dispute between AML and SML is the following. While **S5** is generally thought of as the correct account of metaphysical modality, there are a number of dissenters who have principled reasons for

[10] The following presentation follows that on p.47f of [10]

endorsing weaker logics. For example, according to the combinatorial theory of possibility endorsed by David Armstrong it is argued that the correct modal logic for metaphysical modality is **S4**;[11] similarly, on the basis of arguments concerning the constitution of material objects others have endorsed even as weak a logic as **KT** as the correct logic for metaphysical modality ([14]). In formal semantics there is a methodological pressure to not pre-judge questions as to what the correct metaphysics of a domains is. As a result it is a black mark against the SML framework if, in order to match the expressive capacities of AML, it is committed to the correct logic of metaphysical necessity must be **S5**. To see that this is so consider how one would react to a theory of natural language quantification which required that there be only finitely many objects.

7 Extensions of the Object-Language

This would seem to be bad news for this way of generalising the SML linguistic framework in order to deal with weaker background modal logics. Of course, the proponent of SML could deny the pull of the above methodological considerations, insisting that (as Wehmeier claims in giving the semantics for SML), that the choice of **S5** as the background modal logic "matches the intuitive semantics of ordinary English"[18, p.616]. This does not solve the problem, though, as it will reoccur whenever we add an operator to the object language which the actuality operator doesn't distribute over–that is, any operator which renders the actuality operator non-monocosmic. One simple such extension is the addition to the language of a counterfactual conditional '$\Box\!\!\rightarrow$'. In this setting the above problem will recur as, informally speaking, the introduction of the counterfactual conditional into our language in effect reintroduces accessibility relations into our semantics, albeit formula-relativized ones.

The semantics we are envisaging here is effectively the *selection function* semantics from section 2.6 of [12] where we enrich our models with a function f which takes a formula φ and a world w and yields a set $f(\varphi, w)$ of the closest φ-worlds to w (here we implicitly require that all the worlds y in $f(\varphi, w)$ are ones where φ is true). Of course we can equally well rewrite these semantics so that we instead have, for each formula φ, an accessibility relation R_φ such that $R_\varphi xy$ iff $y \in f(\varphi, x)$—for the sake of familiarity, though, we will stick with the usual notation. Having so extended our models we can define truth at a point in a model for AML-formulas whose principle connective is $\Box\!\!\rightarrow$ as:

$$\mathcal{M} \models_w \varphi \Box\!\!\rightarrow \psi[\sigma] \text{ if and only if } \forall v \in f(\varphi, w) \text{ we have } \mathcal{M} \models_v \psi[\sigma].$$

Similarly, for SML-formulas with $\models$ replaced by $\Vdash$. To see that this causes problems consider the case where we extend the models $\mathcal{M}$ and $\mathcal{N}$ so that

[11] Technically all that is argued on p.62 [1] is that the correct modal logic for metaphysical modality on the combinatorialist account does not prove the **B** axiom (i.e., $p \rightarrow \Box\Diamond p$). This leaves open the (often overlooked) possibility that the correct logic could be one of the many logics between **S4** and **S5**, or a $\subseteq$-incomparable extension of **S4** like **S4M** (for more information on which the reader should consult p.146 of [3], where **M** is referred to as $\mathbf{G}_c$)

there are nullary predicates A and B true, respectively, at w_0 and w_1, and set $f(A, @) = \{w_0\}$ and $f(B, @) = \{w_1\}$. Then we can show that the following formula of QL_A is true in $\mathcal{M}$ and false in $\mathcal{N}$, while the two models are indistinguishable in QL_S—taking both languages to be extended by the inclusion of the counterfactual conditional.

$$A \mathrel{\Box\!\!\rightarrow} \exists x(Fx \wedge \mathbb{A}(B \mathrel{\Box\!\!\rightarrow} (\exists y(Fy \wedge x \neq y)))). \tag{7}$$

In order to bring out the significance of the situation being modelled, consider a case where we have a partition on the space of worlds W, each side of the partition agreeing upon A, and the actual world being an A-falsifying world. Then we might wish to know whether F-ness is had by the same things in the A-worlds as in the not-A-worlds. This question amounts to wondering whether the following formula is true in the model, with $\mathrel{\Diamond\!\!\rightarrow}$ being the dual of $\mathrel{\Box\!\!\rightarrow}$.[12]

$$A \mathrel{\Box\!\!\rightarrow} \exists x(Fx \wedge \mathbb{A}[\neg A \mathrel{\Diamond\!\!\rightarrow} (\exists y(Fy \wedge x \neq y))]).$$

This is a nice example of one of the primary advantages which the actuality operator has over having indicatively and subjunctively marked predicates, namely that the actuality operator allows for 'world-travelling' of a kind which is precluded by merely having marked predicates. In fact this ability to facilitate world-travelling by being able to operate upon arbitrary formulas is one of the main reasons for preferring AML style languages to SML style ones.

Here again SML is expressively weaker than AML. This weakening of the expressive capacities of our language would not be that worrying if it had no impact upon the empirical adequacy of the two frameworks—if the kinds of sentences which could be expressed in one framework in weaker background logics, or by the presence of counterfactual conditionals, were just merely formal artefacts. The case of the above counterfactual conditionals seem to suggest otherwise, though. Consider the following sentence.

(8) If Australia had been a Dutch colony then someone would be Prime Minister who wouldn't have been Prime Minister had Australia been a French colony.

This sentence appears to be one of the following form—the relative similarity measure required by the embedded counterfactual being relative to the actual world, rather than the world(s) which the antecedent directs us to.[13]

$$DC(Australia) \mathrel{\Box\!\!\rightarrow} \exists x(PMx \wedge \mathbb{A}(FC(Australia) \mathrel{\Box\!\!\rightarrow} (\exists y(PMy \wedge x \neq y)))).$$

It is easy to see that this is of the same form as (7), which cannot be expressed in the obvious kind of extension of SML by the addition of $\mathrel{\Box\!\!\rightarrow}$.

[12] Here we are assuming that $f(A, @) = \{w_0\}$ and $f(\neg A, @) = W \setminus f(A, @)$—i.e. $f(\neg A, @) = \{@, w_1\}$.

[13] Sentences such as this one show how careful one must by when talking about the nesting of subjunctive conditionals. This is an example where we have subjunctive conditions embedding under other subjunctive conditionals syntactically in the sense that there is a subjunctive conditional under the scope of another subjunctive conditional, but not semantically in sense that evaluation of the conditional in the antecedent forces us to examine the worlds pertinent to the truth of the antecedent. cf footnote 25 of [21].

8 'Indicative' Modal Operators

One other way which the proponent of SML may try and mitigate the problems caused by the expressive weakness of the framework in weaker logics is to draw the indicative/subjunctive distinction at the level of operators as well as at the level of predicates and quantifiers. In essence this is what Wehmeier does in 'Subjunctivity and Conditionals', treating the subjunctive conditional (which we will now write as '$>$' to facilitate comparison with '$\Box\!\!\rightarrow$' above) so that when evaluating $\varphi > \psi$ at w we check the truth of ψ at the nearest φ worlds to @.

$$\mathcal{M} \Vdash_w \varphi > \psi[\sigma] \text{ if and only if } \forall v \in f(\varphi, @) \text{ we have } \mathcal{M} \Vdash_v \psi[\sigma].$$

With a conditional like '$>$' we can treat sentences like (7) as follows:

$$DC(Australia) > \exists x(PMx \wedge (FC(Australia) > (\exists y(PMy \wedge x \neq y)))).$$

As a treatment of subjunctive conditionals the conditional '$>$' requires that there be no true embeddings of conditionals in the consequents of subjunctive conditionals—e.g. that there be no true sentences which would be represented as $A \Box\!\!\rightarrow (B \Box\!\!\rightarrow C)$ in the $\Box\!\!\rightarrow$-language, otherwise this approach will generate the wrong truth conditions. We will leave that aside for the moment, though, to mention how this kind of approach is thought to help the SML framework deal with expressive difficulties.

In our generalisation of the SML-s framework to weaker modal logics we gave a semantic treatment for the modal operators which uses accessibility relations according to which the truth of a box or diamond formula at a world w depends upon the worlds which are R-accessible from w, as is commonly done in traditional accounts of the weaker modal logics. Generalising the indicative/subjunctive distinction to operators, this involves thinking of the modal operators as being 'subjunctive' in the sense that their truth at a world depends upon goings on at that world. This kind of contrast already needs to be drawn in quantified versions of SML and AML in the form of the indicative and subjunctive quantifiers. Thus the proponent of SML might think of the modal operator as being 'indicative', as discussed briefly above in §3, with the truth in a model of $\Diamond$-formulas in this sense (annotated $\Vdash^i$) being as follows:

$$\mathcal{M} \Vdash^i_w \Diamond A \text{ if and only if } \exists v(R@v \text{ and } \mathcal{M} \Vdash^i_v A).$$

Here we use $\Vdash^i$ to indicate truth at a point in a model on this new construal of the SML framework, which we will call SML-i for 'SML with indicative modal operator'. This semantics for the modal operator is very similar to what is called the 'centred semantics' for the knowledge operator given in [2, p.183f]. In the centred semantics for knowledge, formulas such as $K\varphi$ (as we will write Bonnay and Ègreè's epistemic modal operator) are evaluated relative to a pair of worlds (w, w') where the truth of $K\varphi$ at a pair (w, w') requires that φ be true at all points w'' such that Rww''. This is just like our SML-i $\Box\varphi$, except with the world w being an explicit parameter relative to which we evaluate the truth of formulas, rather than a part of our models. It's also clear that the SML-i formula $\Box\varphi$ is equivalent to the AML formula $\mathbb{A}\Box\varphi$.

What we will now argue is that the SML-i is properly expressively weaker than AML where the background modal logic is weaker than **S5**, much like we did above for SML-s. Unlike the example in §5, though, we will only need to consider the propositional fragment of the two languages here—our models being structures of the form $\langle W, @, R, V\rangle$ where W is a non-empty set (of worlds), @ a world in W (the actual world), $R \subseteq W \times W$ and V a function from the (subjunctive) propositional variables to subsets of W. Consider the frame $\mathfrak{F} = \langle\{0, 1, 2\}, 0, \leq\rangle$, and let $\mathcal{M} = \langle\mathfrak{F}, V\rangle$ and $\mathcal{M}_s = \langle\mathfrak{F}, V_s\rangle$ (the 's' being mnemonic for 'swap') be models on this frame, where V and V_s are as follows.

- $V(p) = \{2\}$.
- $V_s(p) = \{1\}$.

These two models are both distinguishable in the standard propositional actuality language, as recorded below.

Proposition 8.1. *$\mathcal{M} \models_@ \Diamond\Box p$ and $\mathcal{M}_s \not\models_@ \Diamond\Box p$.*

What we will now show, though, is that these two models prove the same SML-i formulas. In order to see why this is so it is helpful to consider a modification of our models (over transitive, reflexive frames), and semantics, which brings out what is distinctive about the indicative modal operators. In evaluating SML-i formulas we can dispense with the accessibility relation R, replacing it instead with the set $X = \{x|R@x\}$, making our models structures of the form $\langle W, @, X, V\rangle$, with the truth in a model conditions for $\Diamond$ formulas in SML-i become:

$$\mathcal{M} \Vdash^i_w \Diamond\varphi \text{ if and only if for some } w' \in X : \mathcal{M} \Vdash^i_{w'} \varphi.$$

We will not use these new models in what follows, but it is easy to see that $\mathcal{M}$ and $\mathcal{M}_s$ validate the same SML-i formulas if one can understand why we can move from standard models with accessibility relations to models like those we've just described.

Proposition 8.2. *Given the models $\mathcal{M}$ and $\mathcal{M}_s$, let f be the function from $\{0, 1, 2\}$ to $\{0, 1, 2\}$ such that: $f(w) = (w + 1)mod2$ if $w > 0$ and $f(w) = w$ otherwise. Then for all formulas φ in the language of propositional SML we have the following:*

$$\mathcal{M} \Vdash^i_w \varphi \textit{ iff } \mathcal{M}_s \Vdash^i_{f(w)} \varphi.$$

Proof. By induction on the complexity of A. The basis case is handled by the fact that $f(V(p)) = V_s(p)$, and the inductive case for $\Diamond$-formulas following from the fact outlined above that the truth of $\Diamond$-formulas only depend upon the set $R(0)$, which is the same in both models. □

Theorem 8.3. *SML-i is properly expressively weaker than AML when the background modal logic is* **S4**.

Proof. Follows from Propositions 8.1 and 8.2, and the fact noted above that the SML-i $\Diamond\varphi$ is just the AML $\mathbb{A}\Diamond\varphi$. $\square$

If the proponent of the SML framework takes the modal operator to be 'indicative' then their framework will still be properly expressively weaker than the AML framework—leaving them in no better position than if they took the modal operator to be 'subjunctive'.[14] The proponent of SML could, of course, take their language to contain both an indicative and subjunctive modal operator, and if they did this they would end up with a language which is expressively equivalent to AML over any framework. Let the SML-is language be the language we get when we enrich the language QL_S by the addition of a new modal operator $\Diamond^i$.[15] Let truth in a model for SML-is be as for SML-s with the following added clause.

$$\mathcal{M}, \sigma \Vdash_w \Diamond^i\varphi \text{ if and only if } \exists w'(R@w' \ \& \ \mathcal{M} \Vdash_{w'} \varphi).$$

Theorem 8.4. *SML-is and AML are expressively equivalent over every background modal logic.*

Proof. We sketch the relevant transformations. Giving an AML formula ψ true in the same models as a given SML-is formula φ is simple: replace $Fa_1 \dots a_n$ with $\mathbb{A}Fa_1 \dots a_n$, $\exists xA$ with $\exists^@ xA$ and $\Diamond^i A$ with $\mathbb{A}\Diamond A$, and then remove all subjunctive markers. To given an equivalent SML-is formula ψ true in the same models as a given AML formula φ we need to replace every modal operator under the direct scope of an actuality operator with its indicative counterpart, and every predication and quantifier *not* under the direct scope of an actuality operator with its subjunctively marked counterpart. $\square$

This provide a general method by which the proponent of SML can keep their language expressively equivalent to any extension of an AML language—simply add both an indicative and subjunctive version of the new piece of modal vocabulary. Of course, this leaves them in the unsatisfying position have having to claim that there is a hidden ambiguity between indicative and subjunctive versions of the modal operators. In many ways this seems to be a less comfortable position than having to posit a distinction between a 'weak' and 'strong' indicative mood (e.g. the distinction between Fa being true at @, and $\mathbb{A}Fa$ being true at @) which proponents of SML claim is a commitment of AML frameworks. At the very least it puts them in the position (much like the AML theorist) of having to give some explanation for this ambiguity.

There is a further possibility which brings out quite naturally the fact that these expressive inadequacies are arising due to the fact that we have traded

[14] Adopting then SML-i framework does not give the SML theorist much freedom concerning what background modal logics are available. The logic in the SML-i language determined by the class of all frames is **K45**, over all serial frames **KD45**, and over all reflexive frames **S5**. So, in fact, the *only* alethic modal logic available to the SML-i theorist is **S5** itself. Thanks to an anonymous referee for pointing this out.

[15] The idea of considering a language with both indicative and subjunctive modal operators is due to unpublished work by Helge Rückert.

in an operator for a class of marked predicates. On this alternative approach, rather than adding a new class of indicative predicates, the proponent of the SML approach can treat all predicates as by default indicative, and add a subjunctivity operator S à la [9]. Here the general idea (which generalises the treatment given by Humberstone) would be to have all formulas by default evaluated indicatively—taking this to mean that not only are predicates not under the scope of an S evaluated at the actual world, but also all modal operators are treated in the SML-i fashion unless they occur under the scope of an S operator, in which case they are treated in the manner described in §3. We will not dwell on the relative merits of this alternative here, though.

9 Conclusion

If we take seriously the idea that our formal semantic theories should not prejudge metaphysical issues, the above results show that the SML theorist is on less solid ground than they might at first appear to be. Weakening the background modal logic presents the SML theorist with a choice between three different ways of fleshing out the SML linguistic framework: (i) take modal operators to be 'subjunctive', (ii) take them to be 'indicative', or (iii) to use a mixed language with both 'indicative' and 'subjunctive' modal operators. All three of these options appear to have their costs, though. Taking option (i) leaves the SML theorist with a language which is not only expressively weaker than AML language (as shown in §4) but if this option is lifted to other kinds of modal operators like subjunctive conditionals then their framework ends up being empirically inadequate (§7). Taking option (ii) the SML theorist still ends up with an expressively weaker language, this time even at the propositional level, but is able to deal with examples of the kind mentioned in §7. Lastly, taking option (iii) the SML theorist ends up with a language which is expressively equivalent to the AML language. To do this, though, the SML theorist ends up having to posit two senses of possibility—an indicative and a subjunctive one. This would seem to, at best, put the SML theorist on even footing with the AML theorist, who is charged with having to posit two kinds of indicatives. Despite this it is clear that the mixed SML-is language is the SML theorists best hope for dealing with weaker background logics, and is deserving of further technical and philosophical investigation.

More generally, these results show a reason to be concerned by arguments in favour of less-comprehensive frameworks on the basis of the presence of monocosmic operators—as whether or not an operator is monocosmic is quite sensitive to features of the object language. Whether there are more general reasons to be concerned about such arguments remains to be seen.

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