Notational Variance and its Variants

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Abstract

What does it take for two logics to be mere notational variants? The present paper proposes a variety of different ways of cashing out notational variance, in particular isolating a constraint on any reasonable account of notational variance which makes plausible that the only kinds of translations which can witness notational variance are what are sometimes called definitional translations.

1 Introduction

Could it be that logical pluralists are simply confused, counting certain logics as distinct when they differ only superficially? Given that natural language does not wear its logical form on its sleeve this is at least an open possibility. Moreover, in at least some cases it does appear that we have pairs of logics which differ only superficially. What kinds of differences between logics are merely superficial, though? Susan Haack proposes the following:

> [S]uppose one were to ask how 'classical logic' is to be demarcated. This is to be done, I have supposed, by reference to its set of theorems and valid inferences. Any system with the same theorems/inferences as, say, *Principia Mathematica*, counts as a formulation, a version, of 'classical logic'. In particular a system which differs from that of PM only in employing a distinct, but intertranslatable, notation – say '&' in place of '.' for conjunction – is only a notational variant of classical logic. (Haack, 1974, p.7)

Haack here gives a suggestion for what kinds of differences between logics are merely superficial, and thus don't make a 'real' difference to which logic is picked out. In particular Haack suggests that at least one way in which logics can differ superficially is by differing *typographically*, or to put things into more congenial terminology, one way for logics to be notational variants is for them to be typographical variants. As we will understand the term here two logics are *notational variants* if and only if the only differences between them are merely superficial. This characterisation is vague, and one of the goals of this paper is to attempt to better understand the kinds of differences between logics which are merely superficial.

Questions of notational variance are, as we will see, paradigmatically questions about translations between logics, and we will approach the question of what kinds of differences between different logics are merely superficial by focusing on different properties concerning translations between logics. To do this we will begin in section 2 by fixing some notation and terminology concerning translations, before in section 3 introducing the notion, which we call 'strict notational variance', which most people probably think of when they think of notational variance. Strict notational variance turns out to be far too strict an account of notational variance to accord with some of our logician's intuitions concerning which logics differ merely superficially, and so in section 4 we start to sketch out a more tolerant account of notational variance, isolating an important 'external' constraint on notational variance which we use in section 5 to show that various logics are not notational variants. We then give our tentative characterisation of this more tolerant version of notational variance in section 6, and close by giving some broader consequences which flow from paying attention to the external constraint which is at the heart of our tolerant account of notational variance. Before doing that, though, let us take a further look at what has been said in the literature on superficial differences between logics and languages.

1.1 Notational Variance and Superficial Difference

In the quotation with which we opened this paper Haack appears to be suggesting that it is at least a sufficient condition for a pair of logics to be notational variants that they be typographical variants. If we regard this as a sufficient condition we end up with something in the vicinity of what in section 3 we call 'strict notational variance'. Prima facie this is far too stringent an account of what it takes for two logics to differ superficially, and definitely won't do for capturing the notion which logical pluralists are interested in— as the $\{\wedge, \neg\}$- and $\{\vee, \neg\}$-fragments of classical truth-functional logic are not strict notational variants, and yet it seems that any difference between these two logics is merely superficial. In subsequent work Haack appears to be aware of this issue, as we can see in the following quotation.

> I shall suggest two accounts of 'the same system', one broader and one narrower, each suitable for certain purposes.

> The narrower sense: L_1 and L_2 are alternative formulations of the same system if they have the same axioms and/or rules of inference once allowance has been made for differences of notation (e.g. replacing '&' by '.') and of primitive constants (e.g. replacing 'p&q' by '$\neg(\neg p \vee \neg q)$').
>
> The broader sense: L_1 and L_2 are alternative formulations of the same system if they have the same theorems and valid inferences once allowance has been made for differences of notation and primitive constants. (Haack, 1978, p.21)

In both Haack's 'narrow' and 'broad' sense of two logics being merely superficially distinct allowance is made not just for mere typographical differences between the underlying languages (essentially our strict notational variance), but also for which connectives are taken as primitive. This more liberal notion, which allows for notational variants to differ in their choice of primitives, is more in the direction of our notion of tolerant notational variance, and as we will see this is plausibly the most liberal notion to satisfy our external constraint. Throughout we will be concerned with accounts more akin to Haack's broader sense of two logics being the same system, there being some prima facie problems with Haack's narrower sense as stated.[1]

Discussions of notational variance are not restricted to philosophical logic, of course. In fact the term has earlier providence in discussions of formal grammar, with Chomsky noting that:

> [g]iven alternative formulations of a theory of grammar, one must first seek to determine how they differ in their empirical consequences, and then try to find ways to compare them in the area of difference. It is easy to be misled into assuming that differently formulated theories actually do differ in

[1]As was first noted in Hiz (1958), there are some general difficulties which one has to be aware of when translating axioms if we wish to have our two proof systems in different languages characterise the same logic, difficulties which mean that as stated Haack's narrower sense of notational variance will not result in different proof systems for the same logic. To take an instructive example of this from Humberstone (2004b, p.396–400), consider the standard proof system for the least normal modal logic **K** in the language with primitive connectives $\rightarrow$, $\neg$ and $\Box$, which results by extending a set of axiom schemata sufficient (with Modus Ponens) for classical propositional logic by all instances of the axiom **K** (=$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$) and closing under the rules of Modus Ponens and Necessitation (which takes us from $\vdash A$ to $\vdash \Box A$). If we translate these axioms into the language with primitive connectives $\rightarrow$, $\neg$ and $\Diamond$, making use of the definition of $\Box A$ as $\neg\Diamond\neg A$, then we end up with a proof system $\mathbf{K}_\Diamond$ in which, as noted in Corollary 3.2 of Humberstone (2004b, p.399), we cannot prove $\Diamond p \rightarrow \Diamond\neg\neg p$ (while we are able to prove $\neg\Box\neg p \rightarrow \neg\Box\neg\neg\neg p$ in **K**). What Haack's criterion misses out (and this is the core moral of Hiz (1958)) is that when we translate between axiom systems we need to make sure that, if we take the definition as a metalinguistic abbreviation, the definition in question is provable in the target system.

empirical consequences, when in fact they are intertranslatable—in a sense, mere notational variants. (Chomsky, 1972, p.69)

Here Chomsky appears to be understanding two grammatical theories to be notational variants whenever they are, in essence, empirically equivalent—i.e. agree in all their empirical consequences. Johnson (2015) is an excellent discussion of this notion of notational variance in linguistics. Similar claims about notational variance are also sometimes made when discussing semantic theories of various kinds, for example in Stojanovic (2007). There it is claimed that minimal versions of contextualist and relativist semantic theories of taste ascriptions are intertranslatable, and thus that 'there is never going to be any properly semantic evidence to cut in favor of the one account over the other' Stojanovic (2007, p.703), as any statement which one semantic theory predicts as coming out true in a given situation, the other will also predict as coming out true—the translation showing us how. This again seems to be suggesting that, at least in some sense, the only differences between these two theories are superficial.

2 Translations & Notational Variance

All of the quotations above characterised notational variance as, in some way, involving two formal languages being in some sense intertranslatable, and in the cases where logics were concerned the adequacy of these translations was tied to particular logics defined over these languages. Before we go on, then, we will first need to settle some notational and terminological issues concerning translations and formal languages. We will largely concern ourselves only with propositional languages (largely keeping any mention of first-order languages to asides). Following Wojcicki (1988, p.14) and others, we will think of a propositional language $\mathcal{L}$ as an absolutely free algebra, the generators of which are a denumerable set $p_1, p_2, p_3, \ldots$ of *propositional variables*, and whose operations are the connectives of that language. Moreover, we will think of the extension of a language $\mathcal{L}$ by the addition of a new n-ary operator # as the absolutely free algebra with the same generators as that of $\mathcal{L}$, and whose operations are those of $\mathcal{L}$ along with a new n-ary operation #, calling such a language $\mathcal{L}^{\#}$.

Throughout we will think of logics as consequence relations. Given a pair of consequence relations $\vdash_1$ and $\vdash_2$ we will say that a *translation* is a total function τ from the language of $\vdash_1$ to the language of $\vdash_2$, and say that τ *faithfully embeds* $\vdash_1$ *into* $\vdash_2$ whenever we have, for all formulas $A_1, \ldots, A_n, B$ in the language of $\vdash_1$ the following condition being satisfied.

$$A_1, \ldots, A_n \vdash_1 B \text{ if and only if } \tau(A_1), \ldots, \tau(A_n) \vdash_2 \tau(B). \tag{1}$$

The word 'faithfully' here records the 'if' direction of this condition. Faithful translations are variously called 'conservative' in Silva et al. (1999) and Feitosa and D'Ottaviano (2001), 'unprovability preserving' in Inoué (1990), and 'exact' in Pelletier and Urquhart (2003). Translations which are not necessarily faithful (and thus only satisfy the 'only if' direction of condition (1)) are usually called simply 'translations', or 'sound translations'. We will be concerned with various constraints on the way in which translations respect the structure of formulas. In particular, given a translation τ from $\mathcal{L}_1$ to $\mathcal{L}_2$, let us say that τ is

- *variable-fixed* if $\tau(p_i) = p_i$, for every propositional variable p_i
- *compositional* if for every primitive n-ary connective $\#$ in the language of $\mathcal{L}_1$ there is a formula $\#^\tau(p_1, \ldots, p_n)$ in the language of $\mathcal{L}_2$ constructed out of n propositional variables such that $\tau(\#(A_1, \ldots, A_n)) = \#^\tau(\tau(A_1), \ldots, \tau(A_n))$.

The labels above are taken from Humberstone (2005). Translations which are compositional and meet the condition that $\tau(p_i) = C(p_i)$ for some formula $C(p)$ containing only the propositional variable p, are called *grammatical* in Epstein (1990). Translations which are both compositional and variable fixed are called *definitional* in Wojcicki (1988, p.69f), and will be of special importance in section 6. These conditions are quite common in the literature, and sometimes are imposed as part of what it means for a function from one language to another to be a translation. Less familiarly, we will say that a pair of translations τ_1, τ_2 from $\mathcal{L}_1$ to $\mathcal{L}_2$ are (weakly) *recursively interdependent* if they are both variable-fixed, and compositional on all connectives $\#$ except for on connectives $\#_1$ and $\#_2$ where $\tau_1(\#_1(A_1, \ldots, A_n)) = \#_1^{\tau_1}(\tau_2(A_1), \ldots, \tau_2(A_n))$ and $\tau_2(\#_2(A_1, \ldots, A_m)) = \#_2^{\tau_2}(\tau_1(A_1), \ldots, \tau_1(A_m))$.[2] Examples of translations which are weakly recursively interdependent can be seen in the translation between Data Logic and the modal logic **S4M** in van Benthem (1986, p.234), and in the translation between AML and SML as presented in French (2013, p.1692). We will see another example of a weakly recursively interdependent translation below. A survey containing further examples of translations meeting all of the various combinations of the conditions given here can be found in Chapter 2 of French (2011).

The idea behind cashing out notational variance in terms of translations is that we can see a translation τ as telling us the manner in which $\vdash_1$ is genuinely similar to $\vdash_2$, with tighter constraints on τ reflecting a more restricted understanding of the kinds of differences which we should regard as being merely su-

[2] This condition is a special case of the more general notion of a set of translations being recursively interdependent discussed in French (2011, p.16).

perficial. So if $\vdash_1$ and $\vdash_2$ are notational variants we would expect there to be translations τ_1 and τ_2 such that τ_1 faithfully embeds $\vdash_1$ into $\vdash_2$, and similarly for τ_2. This gives us some degree of common ground against which to assess the proposals we will consider below.

3 Strict Notational Variance

Whether two languages are notational variants ought not be sensitive to the manner in which their respective languages are formulated. One initial way of bringing out this general idea is to begin with an extremely language-sensitive account of notational variance, the more language insensitive version of this being our notion of strict notational variance.

Let us temporarily think of formulas in our language as strings of symbols, rather than (as in our preferred approach described above) as elements in a particular kind of algebra. One way we might try to characterise notational variance, then, is in terms of symbol-for-symbol replacement—providing a function which uniformly maps each symbol of the one language to a symbol of the other. This is one way of understanding Haack's quotation above, suggesting the 'symbol map' which maps the symbol '.' to '&' or vice versa, and otherwise acts as the identity function on all other symbols. Such a symbol-for-symbol replacement view of the translations which render logics notational variants would render differences between presentations of the same language in Russellian and Polish notation as being substantial, rather than merely superficial. For example, let us consider pure implicational logic. Formulas of the language $\mathcal{L}_R$ are determined as follows, where $\frown$ denotes string concatenation.

- A propositional variable p_i is a formula.
- If A is a formula and B is a formula then '('$^\frown A^\frown \rightarrow {}^\frown B^\frown$')' is a formula.
- Nothing else is a formula

Similarly, the language $\mathcal{L}_P$ is determined as follows.

- A propositional variable p_i is a formula.
- If α is a formula and β is a formula then $C^\frown\alpha^\frown\beta$ is a formula.
- Nothing else is a formula

Now consider the formula Cpp from $\mathcal{L}_P$. What symbol in the language of $\mathcal{L}_R$ could we replace C with that would turn this into a formula of $\mathcal{L}_R$, let alone one which will be interpreted appropriately? By simple inspection we can see that there can be no such symbol-for-symbol replacement, allowing us to conclude that these two languages are not, on this account, notational variants. While Arthur Prior would, perhaps, welcome such a result, this seems otherwise distinctly undesirable. These two languages really are notational variants.

One way of respecting the intuition behind such a view of notational variance, without falling into unseemly language dependence, is to stop thinking in terms of such symbol-for-symbol replacements and instead think in terms of what in Beziau (1999, p.147) are referred to as *language-isomorphisms*. In the present notation a translation τ from $\mathcal{L}_1$ to $\mathcal{L}_2$ is a language isomorphism iff τ is a definitional translation where each $\#^\tau(p_1, \ldots, p_n)$ is of the form $\sharp(p_1, \ldots, p_n)$ for some primitive n-ary connective $\sharp$ in the language of $\mathcal{L}_2$. If we write $\# \mapsto \sharp$ whenever $\#^\tau(p_1, \ldots, p_n) = \sharp(p_1, \ldots, p_n)$, we will also want to enforce the condition that $\# \mapsto \sharp \mapsto \#$, reflecting the idea that the only difference between $\mathcal{L}_1$ and $\mathcal{L}_2$ is that in $\mathcal{L}_1$ we write '#' for what in $\mathcal{L}_2$ we write as '♯'. We can then say two logics $\vdash_1$ and $\vdash_2$ are strict notational variants iff there are language isomorphisms τ_1 and τ_2 which faithfully embed $\vdash_1$ into $\vdash_2$ and $\vdash_2$ into $\vdash_1$, respectively, and additionally satisfy the requirement that $\tau_2(\tau_1(A)) = A$ and $\tau_1(\tau_2(A)) = A$, this latter condition following from the constraint that $\# \mapsto \sharp \mapsto \#$. In Wojcicki (1988, p.67) this notion is offered as an account of 'what it is for two logics to coincide up to notation', strict notational variants there being referred to as 'notational copies' of one another. It is easy to see that the functions τ and τ^{-1}, where $\tau(C\alpha\beta) = (\tau(\alpha) \to \tau(\beta))$, are language isomorphisms between $\mathcal{L}_P$ and $\mathcal{L}_R$, so this notion is at least not quite so sensitive to the manner in which a language is presented.

In many ways cases like the one we have discussed in this section, and that used by Haack, are rather staid. There are some more interesting examples of even this very restrictive notion to hand, though. For example, let $\vdash_1$ be the consequence relation for the single-variable monadic fragment of first-order logic— the first-order language without identity or individual constants with only monadic predicates and a single quantified variable in which, for example, $\forall x Fx \wedge \forall x Gx$ is a formula, while $\forall x \forall y(Gx \wedge Gy)$ and $\forall x \forall y Rxy$ are not—which holds between a set of formulas and a formula if on any variable assignment if all the formulas in the set are satisfied on that variable assignment, then so is the conclusion formula. Let $\vdash_2$ be the so-called *local* consequence relation for the normal modal logic **S5** which requires truth preservation at a point in a universal Kripke-model.[3] In Wajsberg (1933) it is shown that these two languages are strict notational variants, with what is there written as $|A|$ being what we would write as either $\forall x Ax$ or $\Box A$. Technically, as we have defined it above, these two languages are not isomorphic, as any relevant translation would not be variable-fixed. In the

[3] For more on this theme see Porte (1982).

case where languages do not have the same stock of atomic expressions we ought to weaken this requirement to that of translating atomic expressions of the one language (e.g. propositional variables) as atomic expressions of the other (e.g. primitive unary predications). Further, it is interesting to note that in order for these two languages to count as isomorphic we also need to treat '$\forall x$' as a single syntactical unit, rather than as as two (a quantifier which binds a variable). This last consideration points towards an interesting way in which this notion is sensitive to very subtle syntactical differences.

4 Towards a more Tolerant Notational Variance

Strict notational variance does an admirable job of formalising the idea that the only kinds of superficial differences which make for notational variants are typographical differences, where typographical differences are captured by the presence of language isomorphisms. It does a poor job, though, of capturing the common thought that logics which differ only in which sets of interdefinable connectives we take as primitive differ only superficially. This suggests that the kinds of translations which we ought to be concerned with should be more liberal than language isomorphisms. How liberal ought we to be concerning the kinds of translations we are after in order to capture this more liberal notion, though?

Krister Segerberg, in giving an informal account of his notion of 'syntactic equivalence' says that two logics 'come to the same thing' if

> [I]n the first place the languages in which they are formulated are intertranslatable—if what can be expressed in one language can be expressed in the other one, too—and secondly whenever an argument in the one logic is valid, then its counterpart in the other is also valid. (Segerberg, 1982, p.42f)

Segerberg then proposes to make this more precise by proposing the following account of when two logics are syntactically equivalent.

Definition 4.1. *Two logics $\vdash_1$ and $\vdash_2$ are* syntactically equivalent *iff there are translations τ_1 and τ_2 such that τ_1 faithfully embeds $\vdash_1$ into $\vdash_2$, τ_2 faithfully embeds $\vdash_2$ into $\vdash_1$, and in addition we have the following 'inverse' conditions satisfied:*

$$A \dashv\vdash_1 \tau_2(\tau_1(A))$$
$$A \dashv\vdash_2 \tau_1(\tau_2(A))$$

Note that if we restrict ourselves to translations which are language isomorphisms this gives us our definition of strict notational variance above. What is worth noting is that Sergerberg himself formally places no restrictions on the structure of the translations which witness two logics being syntactically equivalent—resulting in a very permissive account of when two logics are notational variants. The work of the rest of this article will be concerned with showing that, while this definition captures much of what we might (speaking metaphorically) think of as capturing the 'internal' or 'language static' aspects of notational variance, it fails to capture what we might think of as its 'external' or 'language dynamic' aspects.

There are various external features of pairs of logics which we might think are relevant to them being notational variants. Having isomorphic spaces of theories is one which is isolated by Caleiro and Gonçalves (2007) as being a core desiderata for two logics 'coming to the same thing'. The main example of such a feature we will be concerned with here has to do with the extent to which matters of notational variance should be insensitive to the addition of new, uniformly treated, vocabulary. In particular, I want to argue that if two logics are notational variants then if we add a new operator with the same intrinsic properties to both of their languages, then the resulting logics should also be notational variants. To give our main motivating example, if two propositional non-modal logics are notational variants, then their modal extensions ought to be notational variants also. Why think this? Because having certain notational variants ought to be an intrinsic property of a logical system, and so should not be disturbed by expanding the language. Such disturbances, when they occur, would seem to suggest that the appearance that two logics are notational variants is dependant on some extrinsic feature of the particular expressive resources of the logics involved, resulting in the putative notational variance being an apparently relational (and thus not intrinsic) matter.

This line of thought suggests the following, somewhat informal, constraint on when two logics are notational variants.

> (External Equivalence) If $\vdash_1$ and $\vdash_2$ are notational variants, then for every way of extending $\vdash_1$ and $\vdash_2$ by the addition of a new operator # with the same properties in both logics should also be notational variants.

How should we understand the idea of two logics both being extended by the addition of a new operator which has the same properties in both logics? Here it is perhaps simplest to shift from thinking about translations and notational variance in the (perhaps somewhat austere) terms we have been so far, with logics considered as consequence relations being our sole object of study, to thinking in more semantic terms.[4] Let us briefly consider the following question: how do we

[4] In discussing *tolerant notational variance* below we will present a more syntactic notion of what

add a normal modal operator to a many-valued logic? By far the most straightforward way is to take the standard possible worlds semantics and, in essence, treat each world as a model for our many-valued logic, i.e. a many-valued valuation. We then interpret our modal operator as being true at a world just in case the formula it is applied to is true at all the accessible worlds.[5] This is to add a (normal) modal operator to a many-valued logic in an analogous manner to how we add them to classical propositional logic. Prima facie this strikes me as a case where we have added a modal operator to a many-valued logic in the same way that we usually add it to classical logic, and with the same intrinsic properties. This heuristic will not, of course, work in all cases, but given that the particular case we are concerned with is one where the logics in question are largely motivated semantically, and appear to not be notational variants when a modal operator is added to their languages, it will do for our present purposes. In order to more fully understand this proposed constraint, though, we would ideally want a fuller understanding of when extensions of a pair of logics treat an introduced piece of vocabulary in the same way.

Let us, then, think of our logics as being given to us semantically in terms of classes of models $\mathsf{Mod}(\vdash)$, where $\Gamma \vdash A$ just in case every model in which all the members of Γ are true is a model in which A is true. In the present case we will mainly be concerned with the case where our classes of models are either boolean valuations or Kripke models. In particular, given a logic $\vdash_i$ determined by a class of boolean valuation V_i, let us then say that the *modal lifting* of $\vdash_i$ is the class of $\mathcal{V}_i$-Kripke models $\langle W, R, V\rangle$, where W is a non-empty set (of worlds), $R \subseteq W \times W$ a binary relation of accessibility, and where V assigns to each world $w \in W$ a $\mathcal{V}_i$-valuation v_w where $v_w(\Diamond A) = T$ iff there is a $w' \in W$ where wRw' and $v_{w'}(A) = T$. The logic $\vdash_i^\Diamond$ determined by the modal lifting of $\vdash_i$ is then the one which holds between a set of formulas $\Gamma \subseteq \mathcal{L}_i^\Diamond$ and formula $A \in \mathcal{L}_i^\Diamond$ just in case, for all $\mathcal{V}_i$-Kripke models $\mathcal{M}$ if all the formulas in Γ are true at a world w in $\mathcal{M}$, then so is A. Against this background, then, let us consider the following condition.

> (Modal Extension) If $\vdash_1$ and $\vdash_2$ are notational variants, then the logics determined by their modal liftings $\vdash_1^\Diamond$ and $\vdash_2^\Diamond$ are also notational variants.

Satisfying (Modal Extension) is, given the above understanding of what it is for an operator to have the same intrinsic properties in two logics, a necessary

it is to extend a pair of logics by an operator which behaves the same way in both languages which relies there on some particular features of tolerant notational variance.

[5] Essentially this way of treating many-valued modal logic is proposed in Priest (2008, p.242), although there we are given a slightly more nuanced treatment of the modal operators than is required here.

condition for satisfying (External Equivalence), and it is on this condition which we will focus in the next section to demonstrate the failure of syntactic equivalence to imply external equivalence. To do that we will make use of a plausible connection between notational variance and expressive equivalence (introduced below). The intuition here is simple, if two logics are notational variants and are determined by classes of models of the same kind (i.e. boolean valuations, or Kripke models), then they ought to be able to say the same things about those models.

This consideration gives rise to the following general principle:

> (Expressivity) If $\vdash_1$ and $\vdash_2$ are notational variants determined by models of the same kind then $\vdash_1$ and $\vdash_2$ are expressively equivalent over those models.

In what follows we will only need to make use of the case where $\mathrm{Mod}(\vdash_1)$ and $\mathrm{Mod}(\vdash_2)$ are classes of Kripke models, but the general point ought to hold for any logics determined by the same kinds of models.

Definition 4.2. *Suppose that $\vdash_1$ and $\vdash_2$ are such that $\mathrm{Mod}(\vdash_1)$ and $\mathrm{Mod}(\vdash_2)$ are both classes of Kripke models. Let us say that $\vdash_1$ and $\vdash_2$ are* expressively equivalent *iff*

$$\forall\varphi \in \mathcal{L}_1 \exists\psi \in \mathcal{L}_2 \forall M \in \mathrm{Mod}(\vdash_1) : M \models_1 \varphi \iff g(M) \models_2 \psi.$$
$$\forall\psi \in \mathcal{L}_2 \exists\varphi \in \mathcal{L}_1 \forall M \in \mathrm{Mod}(\vdash_1) : M \models_1 \varphi \iff g(M) \models_2 \psi.$$

where g is a bijection between $\mathrm{Mod}(\vdash_1)$ and $\mathrm{Mod}(\vdash_2)$ such that $g(\langle W, R, V\rangle) = \langle W, R, V'\rangle$ for some V'.

This definition differs in a number of ways from standard definitions of expressive equivalence, but is analogous to the definitions given in Melia (1992, p.53) and French (2015, p.241).

5 Failures of External Equivalence

The two languages we will primarily be concerned with in this section are the following. Let $\mathcal{L}_1$ be the language constructed out of a denumerable supply of propositional variables $p_1, p_2, p_3, \ldots$ using the unary connective $\neg$ for negation, and the binary connective $\wedge$ of conjunction. Let $\mathcal{L}_2$ be the language constructed out of two-sorts of propositional variables $p_1, p_2, p_3, \ldots$ and $p_1^\perp, p_2^\perp, p_3^\perp, \ldots$ using the boolean connectives $\wedge$ of conjunction and $\vee$ of disjunction. The language $\mathcal{L}_2$ is inspired by the kind of language used in Tait (1968), which draws inspiration from a notational convention due to Schütte. Intuitively we are to think of $p^\perp$ as being the negation of p. Essentially this language is used in Burgess (2009, pp.107–108) in a discussion of how to understanding negation in Anderson & Belnap's logic of first-degree entailment (FDE). Suppose that, given a formula A, we use the de Morgan equivalences to push negations inwards so that they have scope only over propositional variables and then replace all occurrences of $\neg p_i$ with $p_i^\perp$, and call the resulting formula A^*. Then Burgess's idea is that we can say that $A \vdash B$ is FDE-valid iff $A^* \vdash B^*$ is classically valid, where we treat $p^\perp$ as an arbitrary new propositional variable. Here we will be giving $\mathcal{L}_2$ a more 'classical' semantics, interpreting both it and $\mathcal{L}_1$ using boolean valuations: functions from the language to the set $\{T, F\}$. In particular, let us say that a valuation v is:

- $\wedge$-*boolean* iff $v(A \wedge B) = T$ iff $v(A) = T$ and $v(B) = T$
- $\vee$-*boolean* iff $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$
- $\neg$-*boolean* iff $v(\neg A) = T$ iff $v(A) = F$
- *Tait-Schütte* iff $v(p_i) = T$ iff $v(p_i^\perp) = F$

We will interpret $\mathcal{L}_1$ semantically using the class V_1 of $\wedge$- and $\neg$-boolean valuations, and $\mathcal{L}_2$ using the class V_2 of all $\wedge$- and $\vee$-boolean Tait-Schütte valuations. We can then think of $\vdash_1$ as being the consequence relation which holds between a set of formulas $\{A_1, \ldots, A_n\}$ and a formula B from $\mathcal{L}_1$ iff for all $v \in V_1$ whenever $v(A_i) = T$ for all $1 \leqslant i \leqslant n$ we have $v(B) = T$, and similarly for $\vdash_2$ and V_2. It is easy to demonstrate that these two languages are syntactically equivalent according to the following translations.

$$\begin{aligned} \tau_1(p_i) &= p_i \\ \tau_1(A \wedge B) &= \tau_1(A) \wedge \tau_1(B) \\ \tau_1(\neg A) &= \tau^{\mathbf{n}}(A) \end{aligned} \qquad \begin{aligned} \tau^{\mathbf{n}}(p_i) &= p_i^\perp \\ \tau^{\mathbf{n}}(A \wedge B) &= \tau^{\mathbf{n}}(A) \vee \tau^{\mathbf{n}}(B) \\ \tau^{\mathbf{n}}(\neg A) &= \tau_1(A) \end{aligned}$$

$$\begin{aligned} \tau_2(p_i) &= p_i \\ \tau_2(p_i^\perp) &= \neg p_i \\ \tau_2(A \wedge B) &= \tau_2(A) \wedge \tau_2(B) \\ \tau_2(A \vee B) &= \neg(\neg\tau_2(A) \wedge \neg\tau_2(B)) \end{aligned}$$

Given these translations the following is easy to verify.

Theorem 5.1. *$\vdash_1$ and $\vdash_2$ are rendered syntactically equivalent by τ_1 and τ_2.*

If we regard syntactic equivalence as being sufficient for notational variance then the above result tells us that these two logics are notational variants. What we will now show, though, is that these two logics do not satisfy (External Equivalence). To do this we will make use of a modified notion of a bisimulation, modelled on ones used in Hennessy and Milner (1985) and Kurtonina and de Rijke (1997), which preserves the modal extension of $\mathcal{L}_2$.

In particular, let us consider the expansion of both $\mathcal{L}_1$ and $\mathcal{L}_2$ by the addition of a single unary modal operator $\diamond$ for possibility. Models for $\mathcal{L}_1^{\diamond}$ are the modal liftings of $\{\wedge, \neg\}$-boolean valuations, which are easily seen to be equivalent to standard possible worlds models. Models for for $\mathcal{L}_2^{\diamond}$ are the modal liftings of $\{\wedge, \vee\}$-boolean valuations which are also Tait-Schütte valuations, which are equivalent to standard possible worlds models which meet the constraint that $V(p_i) = W \setminus V(p_i^{\perp})$. Let us call such models *Tait-Schütte-models*.

Definition 5.2. *Suppose that* $\mathcal{M} = (M, R, V)$ *and* $\mathcal{N} = (N, S, V')$ *are Tait-Schütte-models, and let* Z *be a binary relation between* M *and* N. *Then* Z *is a* directed diamond simulation *between* $\mathcal{M}$ *and* $\mathcal{N}$ *if it satisfies the following clauses:*

- *If* wZv *then, for all propositional atoms* p_i *and* $p_i^{\perp}$, *if* $\mathcal{M} \models_w p_i$ *(*$\mathcal{M} \models_w p_i^{\perp}$*) then* $\mathcal{N} \models_v p_i$ *(*$\mathcal{N} \models_v p_i^{\perp}$*)*

- *If* wZv *and* Rww' *then* $\exists v'$ *s.t.* Svv' *and* $w'Zv'$.

Note that we could equally well have strengthened the first condition here to a biconditional. This weaker condition is used in Kurtonina and de Rijke (1997, p.2) because they are concerned with languages which lack negation but which have both modal operators. Of particular interest to us is the fact that only adding one of the modal operators to a language which doesn't have negation present as a sentence forming operator leaves us with a language that is expressively weak.

Theorem 5.3. *(Preservation) If there exists a directed diamond simulation* Z *between* $\mathcal{M}$ *and* $\mathcal{N}$, *and* wZv *then, for all formulas* A *in* $\mathcal{L}_2^{\diamond}$, *we have that*

$$\mathcal{M} \models_w A \Rightarrow \mathcal{N} \models_v A.$$

Proof. By induction on the complexity of A.

Basis Case: If $A = p_i$ or $A = p_i^{\perp}$ then this follows directly from the fact that a directed diamond simulation preserves the values of propositional variables.

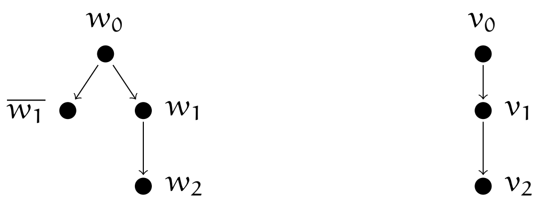

Figure 1: The Frames F_1 and F_2.

Inductive Step:

- CASE 1: $A = B \wedge C$ or $A = B \vee C$. Suppose that $\mathcal{M} \models_w A \wedge B$ and wZv. Then it follows that $\mathcal{M} \models_w A$ and $\mathcal{M} \models_w B$ and so by the induction hypothesis that $\mathcal{N} \models_v A$ and $\mathcal{N} \models_v B$ and hence that $\mathcal{N} \models_v A \wedge B$ as desired. The case for $A = B \vee C$ follows similarly.

- CASE 2: $A = \Diamond B$. Suppose that $\mathcal{M} \models_w \Diamond B$ and wZv. Then for some w' s.t. Rww' we have $\mathcal{M} \models_{w'} B$. So by the definition of a directed diamond simulation we have that there is a $v' \in N$ s.t. Svv' and $w'Zv'$. So by the induction hypothesis it follows that $\mathcal{N} \models_{v'} B$ and, as Svv', that $\mathcal{N} \models_v \Diamond B$, as desired.

□

What we will now show is that these two languages are not expressively equivalent, using an example taken from Hennessy and Milner (1985, p.146). Consider the frames F_1 and F_2 in Figure 1, and let M_1 and M_2 be the models where $V(p_i) = \emptyset$ (and so $V(p_i^{\perp}) = W$). It is easy to show that the following two functions are directed diamond simulations between M_1 and M_2.

- $Z_1 = \{(w_0, v_0), (\overline{w_1}, v_1), (w_1, v_1), (w_2, v_2)\}$

- $Z_2 = \{(v_0, w_0), (v_1, w_1), (v_2, w_2)\}$

So it follows by the Preservation Theorem above that M_1 and M_2 verify the same $\mathcal{L}_2^{\Diamond}$-formulas. These two frames can be distinguished in $\mathcal{L}_1^{\Diamond}$, though, as $\Diamond\neg\Diamond\neg(p \wedge \neg p)$ is true at w_0 in M_1 (as $\overline{w_1}$ is a dead end), while it is false at v_0 in M_2. It follows from this that $\vdash_1^{\Diamond}$ and $\vdash_2^{\Diamond}$ are not expressively equivalent.[6] Thus,

[6] If they were there would have to be a formula $\psi \in L_2^{\Diamond}$ which is true in all models which validate $\Diamond\neg\Diamond\neg(p \wedge \neg p)$. But any model on F_1 validates this formula, and no model on F_2 does. Now

by (Expressivity) we are able to use conclude that $\vdash_1^\Diamond$ and $\vdash_2^\Diamond$ are not notational variants. Finally, then, by (Modal Extension), it follows that $\vdash_1$ and $\vdash_2$ are also not notational variants.

What is going on here? The syntactic equivalence between $\vdash_1$ and $\vdash_2$ relies on the fact that we are able to use the de Morgan equivalences to transform every $\mathcal{L}_1$-formula into a corresponding $\mathcal{L}_2$-formula. In particular we can always push negations inward, replacing connectives with their de Morgan duals, so that they only have scope over propositional variables. By adding a new operator to the language of $\mathcal{L}_2$ without adding its de Morgan dual we end up with the extension of $\vdash_1$ having more expressive power than the corresponding extension of $\vdash_2$. These two languages do satisfy the much weaker condition that for every expansion of $\mathcal{L}_1$ there is an expansion of $\mathcal{L}_2$ such that these expansions are expressively equivalent (just add to $\mathcal{L}_2$ whatever you add to $\mathcal{L}_1$ along with their de Morgan duals). The fact that these two languages satisfy this weaker condition without satisfying our stronger (External Equivalence) points towards the fact that these two languages are more than merely superficially different, with real work being done behind the scenes.

Slight variations on this general pattern of argument can be used to show that a number of modal languages which are expressively equivalent are not notational variants. For example:

- The modal logic **S5** is expressively equivalent to **S5** plus public announcements van Ditmarsch et al. (2008, p.231), but **S5** plus common knowledge is not expressively equivalent to **S5** with public announcements and common knowledge (the latter language being expressively stronger than the former as shown on van Ditmarsch et al. (2008, p.232)). As a result, these two logics are not notational variants.

- The language of modal logic with an actuality operator (AML), and the language of modal logic with indicatively marked and subjunctively marked propositional variables (SML) are expressively equivalent, as shown in Wehmeier (2004) and French (2013). As noted in French (2015), though, these two languages are not expressively equivalent if we add a counterfactual conditional to both languages, and so by (External Equivalence) these two languages are not notational variants. As in the case of the Tait-Schütte language $\mathcal{L}_2$ above, this is also because while AML is 'monocosmic', and thus all occurrences of an actuality operator can be pushed inwards so that they only have scope over propositional variables, this property is broken if we add a standard counterfactual conditional to both languages.

consider the case where $g(M) = M_1$ and $g(M') = M_2$. We know that both M_1 and M_2 must agree on the truth of ψ, from which it follows that M and M' must agree on the truth of $\Diamond\neg\Diamond\neg(p \wedge \neg p)$. But these are both models on F_1 and F_2 respectively.

In both Humberstone (2004a) and French (2015, p.240f), picking up on a general line of argument due to Smiley (1996), it is claimed that

> it is not a good idea to argue against a more comprehensive language ... and against a less comprehensive one, on the basis of an observation that everything that can be said in the richer language has an equivalent in the poorer, since if we work only with the poorer language, we can no longer formulate the observation in question. Humberstone (2004a, p.49)

In the present case the 'observation in question' is the fact that, making use of the de Morgan laws, we can push negations inwards so that they only have scope over propositional variables. The present results give another reason why this is a bad idea—namely that it makes the equivalence of the less comprehensive language (e.g. our $\mathcal{L}_2$) with the more comprehensive one reliant on extrinsic features of the expressive resources available in the language. In this case, for example, the equivalence is reliant on every operator having a de Morgan dual in the language, and so if we add an operator to both languages which is not its own de Morgan dual then the two expanded languages will no longer be equivalent.

6 Tolerant Notational Variance

If we think that (External Equivalence) is a reasonable desiderata for two logics being notational variants, then the results of the previous section make quite clear that notational variance must be more demanding than mere syntactic equivalence. One reasonable way of strengthening syntactic equivalence is to consider *definitional equivalence*. Recalling that a translation is *definitional* if it is both variable-fixed and compositional, let us say that two logics are *definitionally equivalent* if they are rendered syntactically equivalent using definitional translations. Understanding definitional equivalence as being sufficient for notational variance also gives us a nice story about the manner in which definitionally equivalent logics come to the same thing, as when two logics are definitionally equivalent they share a common definitional extension—this is in fact how the notion is first characterised in Wojcicki (1988, p.68). Concerning the notion Wojcicki notes that

> [s]ince all properties of propositional calculi ... but those explicitly related to their language are preserved under definitional extensions ... we shall treat them as linguistic variants of one another. (Wojcicki, 1988, p.66)

The properties Wojcicki had in mind were properties concerning the behaviour of the consequence relation, such as being closed under replacement of logical equivalents, or (as emphasised in Caleiro and Gonçalves (2007)) having isomorphic lattices of theories. There is good reason, though, for thinking that notational variants will satisfy reasonable explications of the (External Equivalence) condition as well. In particular let us take very seriously the idea that the common definitional extension gives, to speak very figuratively, the logical reality underlying the logics which it is a common definitional extension of. Then in order to uniformly extend a pair of definitionally equivalent logics by the addition of a new connective #, we can simply take a (conservative) extension of their common definitional extension and extend them so that they agree with their common definitional extension over their vocabulary extended by #.

More formally, let us restrict our attention to consequence relations which are closed under replacement of logical equivalents. Say that $\vdash'$ is a definitional extension of $\vdash$ iff it is a conservative extension of $\vdash$ whose language extends that of $\vdash$ by new operators $\#_i$, where for each new n-ary operator $\#_i$ there is a formula A constructed out of at most n-propositional variables in the language of $\vdash$ s.t. $\#_i(p_1, \ldots, p_n) \dashv\vdash' A(p_1, \ldots, p_n)$. A consequence relation $\vdash'$ is a common definitional extension of $\vdash_1$ and $\vdash_2$ iff its language contains precisely the connectives of both $\vdash_1$ and $\vdash_2$ and it is a definitional extension of both. Suppose now that $\vdash_1$ and $\vdash_2$ are definitionally equivalent, and thus share a common definitional extension $\vdash_3$, and suppose further that we have a set of sequents $S_\#$ containing a new connective # in the language of $L_3^\#$ and let $\vdash_3^{S_\#}$ be the least conservative extension of $\vdash_3$ by the sequents in $S_\#$ which is closed under replacement of logical equivalents. Let $\vdash_i^{S_\#}$ be the reduct of $\vdash_3^{S}$ to the language $L_i^\#$. It is easy to see that $\vdash_i^{S_\#}$ is a conservative extension of $\vdash_i$, and that $\vdash_3^{S_\#}$ is a conservative extension of $\vdash_i^{S_\#}$. From this it follows trivially that $\vdash_1^{S_\#}$ and $\vdash_2^{S_\#}$ are definitionally equivalent.

Requiring definitional equivalence for notational variance already separates out as genuinely distinct a wide range of logics. To give one example, Wojcicki proves that there can be no faithful definitional embedding of intuitionsitic logic ($\vdash_{IL}$) into classical logic ($\vdash_{CL}$). To see this, recall that Intuitionistic logic has infinitely many non-equivalent formulas in its single-variable fragment, while classical logic has only finitely many such formulas. By the pigeonhole principle, then, any putative translation τ from intuitionistic logic into classical logic must fail to be faithful, as there will be formulas A_i and A_j for which we do not have $A_i \dashv\vdash_{IL} A_j$ but for which we have $\tau(A_i) \dashv\vdash_{CL} \tau(A_j)$, as any definitional translation will translate a formula with a single variable in one language into a formula in a single variable in the other.

Relatedly, in Meyer (1974, p.226–228) it is shown that no system of relevant entailment can be faithfully embedded into any system of modal logic by a definitional translation which translates relevant implication in terms of a modalised truth-function. While this does not show that there is *no* translation of such systems available, it is definitely indicative of such. A similar argument to that above does tell us that the implicational fragment of the relevant logic R is not the strict implicational fragment of the modal logic **S4**, as it is shown in Meyer (1970) that R has 6 equivalence-classes of formulas in its single-variable fragment, while the strict implicational fragment of **S4** has 9, as shown in Byrd (1976). As such there can be no faithful definitional translation from the implicational fragment of **S4** into the relevant logic R.

As an account of notational variance, definitional equivalence seems to bring with it a number of desirable properties, while being a more permissive notion that strict notational variance. Let us say, then, that two logics are *tolerant notational variants* iff they are definitionally equivalent. The results of the previous two sections, then, can be seen as arguing in favour of thinking of tolerant notational variance as being the correct account of notational variance, given the requirement of (External Equivalence).

7 Conclusions & Consequences

Let us take stock. There are a variety of different kinds of notational variance which one can extract from the few places where the notion is discussed in the literature. Which notion of notational variance we ought to concern ourselves with depends on what we want a notion of notational variance to tell us. Concerns with mere typographical difference between languages pushes us towards working with strict notational variance, while attempting to accord with the intuition that truth-functionally-complete fragments of classical logic are all mere notational variants pushes us towards working with tolerant notational variance.

Moreover, isolating tolerant notational variance, and in particular the (External Equivalence) condition, sheds some light on other general discussions where notions in the vicinity of notational variance have been deployed in the philosophical literature. For example, if verbal disagreement between proponents of different formal/logical theories requires that the theories in question be notational variants (which is implicitly suggested in Williamson (2013, p.368)) then many disputes in the literature which have been taken to be merely verbal are in fact clearly substantive because the languages in question fail to be externally equivalent.

- First-order necessitist and contingentist (or actualist and possibilist) languages are expressively equivalent, as noted in Williamson (2013), but fail to be expressively equivalent when, for example, generalised quantifiers are added to both languages, as shown in Fritz (2013, p.657).

- Nihilist and Universalist mereological languages are prima facie notational variants, Warren (2015, pp.428–252) showing that the logical theories most obviously corresponding to these two metaphysical theories are relatively interpretable. As Warren notes, though, this equivalence result breaks down if the resources of plural quantification are added to both languages (with no effect in the case of the nihilist language, as it already has plural quantifiers). As a result, nihilist and universalist mereological languages are not externally equivalent, and thus fail to be notational variants.

Much like the Schütte language we described above, these languages are equivalent if we allow for non-uniform addition of expressive resources. This seems to suggest that there is some kind of substantive (or at least non-notational) difference between the languages in question, but might be taken to merely count against the requirement that verbal disputes require notational variance. Even if this is the case, paying attention to the various variations on notational variance can shed light on a variety of disputes in philosophical logic, broadly construed.

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