Non-Triviality Done Proof-Theoretically

Rohan French Shawn Standefer

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1 Introduction

It is well known that naive theories of truth based on the three-valued schemes K3 and LP are non-trivial. This is shown by the fixed-point model construction of Kripke (1975). Kremer (1988) presents sequent systems for some fixed-point theories of truth, proves a completeness result, and provides an inferentialist interpretation of these systems. Kremer's model constructions show that the systems are non-trivial. Yet, there has been little work done to obtain a proof-theoretic explanation for why these systems are non-trivial, whereas a similar *classical* system is trivial.

Our goal is to gain some insight into *why* systems like K3 and LP are, when endowed with a truth predicate and enough syntactic machinery to get us into trouble, non-trivial by examining how to prove this in a purely proof-theoretic manner—attending to sequent calculi formulations of the two systems. For the sake of simplicity we focus on K3 for the bulk of the paper; the considerations for LP are largely dual. We begin with the basic sequent system and some results and problems in §2. These problems motivate a different sequent formulation in §3, which we show to be non-trivial. We then close in §4 with directions for future work.

2 Opening Moves

The kind of the sequent calculus we have in mind is found in figure 1. Throughout, $\pm A$ stands for A or $\neg A$, depending on whether the sign is $+$ or $-$. We note that to obtain a sequent calculus for LP, one would replace the axiom [K3] with [LP].

$$\frac{}{\Gamma \succ \Delta, p, \neg p}\ [\mathrm{LP}]$$

We will focus on the quantifier-free fragment that includes no variables and no function symbols. There are two reasons. First, the addition of quantifiers changes little with respect to the question with which we are concerned, namely why are certain naive truth theories non-trivial. There is little interaction between the quantifiers and the syntactic theory that we will adopt, namely quotation names. Quantifiers will

$$\frac{}{\Gamma, \pm p \succ \pm p, \Delta}\ [\mathrm{Id}] \qquad \frac{}{\Gamma, p, \neg p \succ \Delta}\ [\mathrm{K3}]$$

$$\frac{\Gamma \succ \Delta, A \quad \Gamma, A \succ \Delta}{\Gamma \succ \Delta}\ [\mathrm{Cut}]$$

$$\frac{\Gamma, A \succ \Delta}{\Gamma, \neg\neg A \succ \Delta}\ [\mathrm{DNL}] \qquad \frac{\Gamma \succ A, \Delta}{\Gamma \succ \neg\neg A, \Delta}\ [\mathrm{DNR}]$$

$$\frac{\Gamma A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta}\ [\wedge \mathrm{L}] \qquad \frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \wedge B, \Delta}\ [\wedge \mathrm{R}]$$

$$\frac{\Gamma, \neg A \succ \Delta \quad \Gamma, \neg B \succ \Delta}{\Gamma, \neg(A \wedge B) \succ \Delta}\ [\neg\wedge \mathrm{L}] \qquad \frac{\Gamma \succ \neg A, \neg B, \Delta}{\Gamma \succ \neg(A \wedge B), \Delta}\ [\neg\wedge \mathrm{R}]$$

$$\frac{\Gamma, A \succ \Delta}{\Gamma, T\langle A \rangle \succ \Delta}\ [\mathrm{TL}] \qquad \frac{\Gamma \succ \Delta, A}{\Gamma \succ \Delta, T\langle A \rangle}\ [\mathrm{TR}]$$

$$\frac{\Gamma, \neg A \succ \Delta}{\Gamma, \neg T\langle A \rangle \succ \Delta}\ [\neg \mathrm{TL}] \qquad \frac{\Gamma \succ \Delta, \neg A}{\Gamma \succ \Delta, \neg T\langle A \rangle}\ [\neg \mathrm{TR}]$$

Figure 1: A Sequent Calculus for K3

be more involved with a more complex syntactic theory, such as classical Peano arithmetic, and further issues will be raised there, such as ω-consistency. The second reason is that quantifiers bring some additional complexity that we think detracts from the overall aim of this project, getting a proof-theoretic grip on non-triviality. We expect that the arguments that we develop can be adapted to include quantifiers with standard rules.

Proposition 1. *Suppose that the sequent $\succ\Delta$ is provable in the above sequent calculus. Then there are some formulas Δ', where each formula $A \in \Delta'$ is a subformula or a negated subformula of a formula in Δ such that $\succ\Delta'$.*

Proof. By induction on the construction of derivations, noting that (i) each of the right-rules have the feature in question, and (ii) that there are no rules which move formulas from the antecedant to the succedent of sequents. □ □

For the moment, we will say that a system is *trivial* if the sequent $\emptyset \succ \emptyset$ is derivable. Since the systems under consideration will have admissible weakening rules, all sequents will be derivable in a trivial proof system. A system is *non-trivial* just in case it is not trivial.

Theorem 2. *The above system is non-trivial.*

Proof. Suppose that we can derive $\emptyset \succ \emptyset$. By inspection of the above rules it is easy to see that it must be that the final rule in such a derivation was [Cut], which means that we have, for some A, (i) $A \succ$, and (ii) $\succ A$. By the above Proposition and (ii) it follows that we have some instance of one of our initial sequents of the form

$$\overline{\succ A_1, \ldots, A_n}$$

for some formulas $A_1, \ldots, A_n$. This is impossible, though, so there is no such derivation. □ □

The basic idea of the consistency proof is simple. In order for $\emptyset \succ \emptyset$ to be derivable, we need the sequents $\emptyset \succ A$ and $A \succ \emptyset$ to be derivable. The sequents are unbalanced, in the sense that they each have an empty side. Either the antecedents or the succedents have been cleared out. If one can show that one cannot derive sequents that are unbalanced on each side, then non-triviality follows. A similar argument can be run for LP, changing Δ and Δ' to be in antecedent, rather than succedent position in Proposition 1 and adjusting the argument in the proof of Theorem 2 accordingly.

The system, as it stands, does not incorporate any syntactic theory. As a result, equivalences, such as a liar Ta being equivalent to $\neg Ta$, or even the weaker co-entailments, are not derivable. For the syntactic theory, we will use quotation names.[1] We start with a classical base language L and extend it to a language, L^+ with truth and quotation names, $\langle A \rangle$, for each sentence A. The names and formulas of L^+ are defined by simultaneous induction, so that $T\langle Fb \rangle$ is a formula, as is $T\langle T\langle Gc \rangle \rangle$. The use of quotation names does not force there to be, for example, liar sentences in the language, although we will assume there are such. To use the syntactic theory, we require that there are new axioms or rules in the language for quotation names and identity. The rules we would like to add are in figure 2.[2] These rules, however, will encounter serious difficulties that will lead us to proceed in a different direction.

$$\frac{\langle A \rangle \neq \langle B \rangle, \Gamma \succ \Delta}{\Gamma \succ \Delta}\ [QNL] \qquad \frac{\Gamma \succ \Delta, \langle A \rangle = \langle B \rangle}{\Gamma \succ \Delta}\ [QNR]$$

In [QNR] and [QNL], $A \neq B$.

$$\frac{\Gamma, s = t, A(t), A(s) \succ \Delta}{\Gamma, s = t, A(t) \succ \Delta}\ [{=}L1] \qquad \frac{\Gamma, s = t \succ \Delta, A(t), A(s)}{\Gamma, s = t, \succ \Delta, A(t)}\ [{=}L2]$$

$$\frac{\Gamma, A(t), A(s) \succ \Delta, s \neq t}{\Gamma, A(t) \succ \Delta, s \neq t}\ [{\neq}R1] \qquad \frac{\Gamma \succ \Delta, A(t), A(s), s \neq t}{\Gamma, \succ \Delta, A(t), s \neq t}\ [{\neq}R2]$$

$$\frac{a = a, \Gamma \succ \Delta}{\Gamma \succ \Delta}\ [{=}Ref] \qquad \frac{\Gamma \succ \Delta, a \neq a}{\Gamma \succ \Delta}\ [{\neq}Ref]$$

Figure 2: Identity and quotation name rules for K3

Identity is not treated partially, as truth is. The classical treatment of identity extends to identities between terms not in the base language. More generally, we also

[1] See Gupta (1982), Kremer (1988), or Ripley (2012).

[2] These two rules are based on those from Kremer (1988) and from Negri and von Plato (2001).

want to consider languages that have predicates in addition to the truth predicate and identity. These base language expressions, including identity, are, in the K3T theory, treated classically. To accommodate them in the sequent calculus, we can add the following axioms, where p is a T-free atom.[3]

$$\frac{}{\Gamma \succ p, \neg p, \Delta}\ [Cl]$$

An alternative, which we will not adopt, is to use the following rule.

$$\frac{\Gamma, p \succ \Delta \quad \Gamma, \neg p \succ \Delta}{\Gamma \succ \Delta}\ [Cl]$$

Let us call the system with the axiom [Cl] and the rules above $K3TL^{=}$, where L is the classical base language. The addition of [Cl] adds another way to obtain derivable sequents of the form $\succ A_1, \ldots, A_n$. This, along with the identity rules that delete formulas raise problems for extending the balance argument in Theorem 2 to work for the extended system.

With identity in the system, the definition of triviality has to be modified, since it may be the case, for example, that while $\emptyset \succ \emptyset$ isn't derivable, $a = \langle \neg Ta \rangle \succ \emptyset$ is.[4] A more general definition of triviality is needed. A system is *trivial*, in the extended sense, if $\Gamma_0 \succ \Delta_0$ is derivable, where Γ_0 is a multiset of equalities, Δ_0 is a multiset of inequalities, and at least one contains a formula with a quotation name on one side and a non-quotation name on the other. Apart from the new definition of triviality, the addition of the identity rules requires modifying the non-triviality proof, since sequents of the form $\succ A_1, \ldots, A_n$ are now derivable. Apart from the preceding issue, the balance argument does not seem to guarantee that the system isn't trivial in the extended sense. While it guarantees that both sides do not end up empty, it does not appear to guarantee non-triviality in the extended sense.

We would like to prove the system $K3TL^{=}$ non-trivial, but the proof-theoretic methods for doing so run into difficulties. We have not seen these pointed out before, so we will briefly indicate some of the hurdles. One way to prove the non-triviality of $K3TL^{=}$ would be to give a cut elimination argument. For a sequent calculus with the structural rules of contraction and weakening absorbed, a common route to eliminating cuts proceeds via a few lemmas, including the *inversion lemma* showing that all the rules are invertible, which says, roughly, that if a derivable sequent could be the conclusion of a rule, then the premiss of that rule is also derivable.[5] In particular, it would require that if $T\langle A \rangle, \Gamma \succ \Delta$ is derivable in n steps, then $A, \Gamma \succ \Delta$ is also derivable in n steps. Our diagnosis of the problem arises is the following: atoms using the truth predicate can occur as the conclusions of rules and as atoms in axioms. Inversion requires that for arbitrary A, $\Gamma, A \succ T\langle A \rangle, \Delta$ be derivable in 1 step, since $\Gamma, T\langle A \rangle \succ T\langle A \rangle, \Delta$ is. But, that is not generally the case.

One might try to force the inversion lemma, by adding as an axiom $\Gamma, A \succ T\langle A \rangle, \Delta$, but this, in turn, would require that an arbitrary A could be inverted. For example, if

[3] The appropriate [Cl] axiom for LP would be $\Gamma, p, \neg p \succ \Delta$.

[4] This problem was pointed out by Kremer (1988).

[5] Negri and von Plato (2001, 32)

A is $B \vee C$, then $\Gamma, B \succ T\langle A\rangle, \Delta$ and $\Gamma, C \succ T\langle A\rangle, \Delta$ would have to be axioms. It appears, then, that obtaining the inversion lemma for these rules would require the addition of a rule with no premises that allows one to infer any derivable sequent. While that would make derivations shorter, it would not be insightful, even if the rest of the argument for the cut elimination theorem worked.

There is one additional hurdle for giving a direct cut elimination argument that we will highlight, as it is important for the approach adopted in the next section. In this style of sequent system, there is usually only an identity substitution rule on the left, in our case the [$=$ L1] and [$\neq$ R1] rules. Substitution on the right is achieved indirectly, starting with the desired term in a formula on the right, e.g. Fb and then proceeding to replace it on the left by means of the rule to obtain, e.g. $a = b, Fa \succ Fb$. This appears, however, to be inadequate in the case of truth as the truth rules can introduce distinguished terms, namely quotation names, in either the antecedent or succedent. None of the other rules introduce terms that are new to the proof. The use of identity substitution rules on the right, as well as the left, creates the same sort of trouble for the elimination proof that truth case did above. There appears, then, not to be any way to obtain $a = \langle Fb\rangle, Fb \succ Ta$ without using cut, as the truth rule would yield $T\langle Fb\rangle$ in the succedent, requiring the use of a term substitution. We will, then, move to a different setting for proving non-triviality.

3 Sequent systems with annotations

We will use an alternative sequent system to deal with some of the indicated issues related to truth and quotation names. Specifically, we will use a modified form of K3TL, without identity rules and without identity in the object language. Suppose one is given a language L^+. Let a *syntax set* $\mathscr{E}$ for L^+ be a set of identities, each of which is of one of the following three forms: $\langle A\rangle = \langle B\rangle$, $a = \langle B\rangle$, or $b = c$, where a, b, c are names that are not quotation names.[6] An *identity set* $\mathscr{E}$ is a syntax set obeying the following closure conditions:

1. for all sentences A, $\langle A\rangle = \langle A\rangle$ is in $\mathscr{E}$,
2. if $s = t$ is in $\mathscr{E}$, then $t = s$ is in $\mathscr{E}$, and
3. if $s = t$ and $t = u$ are in $\mathscr{E}$, then $s = u$ is in $\mathscr{E}$.

Finally, an *annotation set* $\mathscr{E}$ is an identity set not containing $\langle A\rangle = \langle B\rangle$, where A and B are distinct formulas. Given an annotation set $\mathscr{E}$, say that two terms, s, t, appearing in identities in $\mathscr{E}$ are equivalent in $\mathscr{E}$ just in case $s = t$ is in $\mathscr{E}$. We will consider the proof systems K3TL$\mathscr{E}$, for each $\mathscr{E}$. So, a given proof will have a particular annotation set $\mathscr{E}$ that affects its rules.

Taking the interpretations of the identities in an annotation set $\mathscr{E}$ to be fixed to a standard interpretation, then, $\mathscr{E}$ contains the syntactic information of L^+. A particular ground model for L^+ may interpret the non-quotation names, and names outside of $\mathscr{E}$ differently, as well as the predicates, but that is fine, since we are only interested

[6] Note that we are assuming there are no function symbols in the language.

in the syntactic theory, as captured by $\mathscr{E}$. From an inferentialist point of view, the use of the annotation sets presents no philosophical problems.

Rather than use multisets, the systems K3TL$\mathscr{E}$ will use sequences on either side of the turnstile. The sequences Γ, Δ are permitted to be empty. We also add a permutation rule for both sides, although we will generally suppress it in what follows. The purpose of the switch from multisets to sequences is to facilitate the definition of a trace and an ancestor, which are used for the proof of the elimination theorem.

The next definition is used to integrate $\mathscr{E}$ into the proof system. Say that two formulas A and B are equivalent in $\mathscr{E}$ just in case there are sequences of terms $c_1, \ldots, c_n$, and $d_1, \ldots, d_n$, not occurring in quotation names in A, such that B can be obtained from A by replacing one or more occurrences of c_i in A with d_i, where for each i, c_i and d_i are equivalent in $\mathscr{E}$.

The axioms, [Id] and [K3], are generalized to include the following instances, where $\pm p$ is of the form $\pm Tb$. In [K3], the antecedent formulas may be in any order.

$$\frac{}{\Gamma, \pm Tb, \Sigma \succ \Theta, \pm Tc, \Delta}\ [\mathrm{Id}] \qquad \frac{}{\Gamma, Tb, \neg Tc, \Sigma \succ \Delta}\ [\mathrm{K3}]$$

In these axioms, b and c must be equivalent in $\mathscr{E}$. The axiom form of [Cl] does not need to be changed.

The truth rules are similarly modified.

$$\frac{\Gamma, A, \Sigma \succ \Delta}{\Gamma, Tb, \Sigma \succ \Delta}\ [\mathrm{TL}] \qquad \frac{\Gamma \succ \Delta, A, \Sigma}{\Gamma \succ \Delta, Tb, \Sigma}\ [\mathrm{TR}]$$

$$\frac{\Gamma, \neg A, \Sigma \succ \Delta}{\Gamma, \neg Tb, \Sigma \succ \Delta}\ [\neg\mathrm{TL}] \qquad \frac{\Gamma \succ \Delta, \neg A, \Sigma}{\Gamma \succ \Delta, \neg Tb, \Sigma}\ [\neg\mathrm{TR}]$$

In these rules, b and $\langle A \rangle$ must be equivalent in $\mathscr{E}$.

We add the following rules to K3TL$\mathscr{E}$.

$$\frac{\Gamma, A, A, \Sigma \succ \Delta}{\Gamma, A, \Sigma \succ \Delta}\ [WL] \qquad \frac{\Gamma \succ \Delta, A, A, \Sigma}{\Gamma \succ \Delta, A, \Sigma}\ [WR]$$

$$\frac{\Gamma, A, B, \Sigma \succ \Delta}{\Gamma, B, A, \Sigma \succ \Delta}\ [\mathrm{CL}] \qquad \frac{\Gamma \succ \Delta, A, B, \Sigma}{\Gamma \succ \Delta, B, A, \Sigma}\ [\mathrm{CR}]$$

Finally, K3TL$\mathscr{E}$ does not take the rule [Cut] as primitive, although, as we will show, this does not affect what sequents are provable.

An upshot of internalizing the syntactic theory, in the manner that we have done, is that it permits us to return to the simple definition of triviality, namely the derivability of $\emptyset \succ \emptyset$. The reason that we had to move to a more complicated definition of triviality in §2 was that we wanted to permit the use of syntactic resources in the derivation of triviality, as the syntactic resources, in a sense, come for free. The use of any syntactic resources, however, would preclude the derivation of $\emptyset \succ \emptyset$. Since we no longer record appeal to syntactic resources with identities and negations of identities in sequents, the complications are no longer needed.

We will state two lemmas concerning equivalence in $\mathscr{E}$, to be used later.

Lemma 3. *If* A *and* A' *are equivalent in* $\mathscr{E}$ *and* B *and* B' *are equivalent in* $\mathscr{E}$*, then the following are equivalent in* $\mathscr{E}$.

- $\neg A$ *and* $\neg A'$
- $A \wedge B$ *and* $A' \wedge B'$
- $\neg(A \wedge B)$ *and* $\neg(A' \wedge B')$

Proof. This is proved by induction on the complexity of A and B. $\square$ $\square$

It is not the case that if A and B are equivalent in $\mathscr{E}$ then $T\langle A\rangle$ and $T\langle B\rangle$ will be. This is because A and B may be distinct sentences, in which case, one will not have $\langle A\rangle = \langle B\rangle$ in $\mathscr{E}$. This is, however, as it should be. We can say something about relations between formulas equivalent in $\mathscr{E}$.

Lemma 4. *Suppose A and A' are equivalent in $\mathscr{E}$. Then, if $\Gamma \succ \Delta, A, \Sigma$ is derivable, then $\Gamma \succ \Delta, A', \Sigma$ is derivable, and if $\Gamma, A, \Sigma \succ \Delta$ is derivable, then $\Gamma, A', \Sigma \succ \Delta$ is derivable. Furthermore, if the original sequent was derivable in n steps, then the new sequent is derivable in at most n steps.*

Proof. The proof is by induction on the construction of the proof. If A is principle in an axiom, then the result of replacing A with A' in the axiom will still be an axiom, and similarly if A is parametric.

The structural rules are taken care of by the induction hypothesis. We will present [WL], [WR] being similar. Suppose $\Gamma, A, A, \Sigma \succ \Delta$ is derivable. By the inductive hypothesis, then $\Gamma, A', A', \Sigma \succ \Delta$ is derivable. By [WL], $\Gamma, A', \Sigma \succ \Delta$ is derivable. The connective rules are immediate from the induction hypothesis.

Let us look at the truth rules. If A is principle in one of the truth rules, then it is of the form $\pm Tb$. Since A' is equivalent in $\mathscr{E}$, then A' is of the form $\pm Tc$ and c and b are equivalent in $\mathscr{E}$. It follows that the sequent replacing A with A' is also a conclusion of a truth rule. $\square$

$\square$

The contraction rule we use does not permit contraction across formulas equivalent in $\mathscr{E}$. This is, however, shown to be admissible by the previous proof.

Corollary 5. *Fix $\mathscr{E}$ and let A and A' be equivalent in $\mathscr{E}$. If $\Gamma, A, A', \Sigma \succ \Delta$ is derivable, then so is $\Gamma, A, \Sigma \succ \Delta$. If $\Gamma \succ \Delta, A, A', \Sigma$ is derivable, then so is $\Gamma \succ \Delta, A, \Sigma$.*

The form the cut rule that we will show is admissible is the following.

$$\frac{\Gamma \succ \Delta, M, \Xi \quad \Phi, M', \Sigma \succ \Theta}{\Phi, \Gamma, \Sigma \succ \Delta, \Theta, \Xi}\ [\mathrm{Cut}]$$

In this rule, M and M' must be equivalent in $\mathscr{E}$. In light of lemma 3, we could require that M is identical to M', but we will not do so here.

We need to define the notions of being parametric in a rule and parametric ancestor. We use the definitions of Bimbó (2014, 34-35) modified in the obvious way for our rules, which for reasons of space we will leave slightly informal here. The non-displayed formulas in the axioms are parametric in the axioms. In the connective and structural rules, the non-displayed formulas are parametric. A formula occurrence A in the premiss of a rule is a *parametric ancestor* of a formula occurrence B in the conclusion of that rule iff they are related by the transitive closure of the following relation:

Both are occurrences of the same formula and either (i) they are both parametric in the rule and occur in the same position, (ii) they are displayed in [CL] or [CR] and not in the same position, or (iii) they are displayed in [WL] or [WR]. Note that in some rules, such as [$\neg \wedge$ L] and [WR], formulas in the conclusion can have more than one parametric ancestor in the premises, e.g. each formula in Γ in [$\neg \wedge$ L] and A in [WR]. The *contraction count* of a formula occurrence is the number of applications of [WL] or [WR] in which one of its parametric ancestors is displayed. The contraction count of an application of cut is the sum of the contraction counts of the two occurrences of the cut formula.

Define the *trace tree* of an occurrence of a formula as follows. If A is parametric in an inference, then A's trace is extended with a branch containing the corresponding occurrence of A in the premises. If A is principle in a $\neg$ rule, so is of the form $\neg\neg B$, then its trace tree is extended with a branch containing B. If A is principle in a $\wedge$ rule, then it is of the form $B \wedge C$ and its trace tree is extended with a branch containing B and one containing C. If A is principle in a $\neg\wedge$ rule, then it is of the form $\neg(B \wedge C)$ and its trace tree is extended with a branch containing $\neg B$ and one containing $\neg C$. If A is principle in a truth rule, then it is of the form $T b$, and its trace is the displayed B in the premiss. The negated truth rules are similar. If A is principle in a contraction rule, then its trace tree is extended with a branch containing one occurrence of A and one containing the other.

Define the *trace weight* $t(A)$ of an occurrence of a formula A as the number of truth rules featuring nodes of A's trace tree as principle in their conclusions. Define the *grade* $g(A)$ of A as the number of logical connectives appearing in A outside the scope of quotation names. Define the *complexity* of a cut as being $\omega \cdot (t(M) + t(M')) + g(M)$, where M and M′ are, respectively, the occurrences of M and M′ displayed in the left and right premises of [Cut]. The *rank* of the cut is defined as the sum of the *left rank*, which is the number of steps in which a parametric ancestor of M occurs in the succedent of the left premiss, plus the *right rank*, which is the number of steps in which a parametric ancestor of M′ occurs in the antecedent of the right premiss. We will follow Bimbó's triple induction proof technique, modified in the indicated ways to account for the differing rules.[7]

Proposition 6. *Let $\mathscr{E}$ be an annotation set. If $\Gamma \succ \Delta, M, \Xi$ and $\Phi, M', \Sigma \succ \Theta$ are derivable, where M, M' are equivalent in $\mathscr{E}$, then $\Phi, \Gamma, \Sigma \succ \Delta, \Theta, \Xi$ is derivable without cut.*

Proof. It is sufficient to show that uppermost cuts can be eliminated from derivations. The proof proceeds by triple induction on the cut complexity, rank, and contraction count. The left rank is lowered, and then the right rank is lowered, then the complexity is lowered, lowering the contraction count as needed. As usual, we can break the cases into groups, depending on the rank of the left cut premiss and the rank of the right cut premiss. We will present a selection of the cases, generally presenting instances where the cuts are simpler than the general case due to the ordering of formulas.

We will start with the cases in which both cut premises come via axioms.

Case: Both premises are from [Id]. This splits into subcases, depending on whether either cut formula is parametric in the axiom. In the case in which both cut formulas

[7] See Bimbó (2014, 36-15) for details.

are principle, we may have one premiss as $\Gamma, Ta \succ Tb, \Delta$ and the other as $\Sigma, Tc \succ Td, \Theta$. Since we know that the following pairs are equivalent in $\mathscr{E}$, $\langle a, b\rangle$, $\langle c, d\rangle$, and $\langle b, c\rangle$, it follows that a and d are equivalent in $\mathscr{E}$. This means that the sequent $\Gamma, \Sigma, Ta \succ Td, \Delta, \Theta$ is an axiom.

The case in which one premiss comes from [K3] and one from [Cl] is straightforward. Similarly, the case in which one comes from [Id] and one from either [K3] or [Cl] is straightforward.

The permutation cases are taken care of by the induction hypothesis on rank.

Case: The left premiss comes via [*W*R]. This breaks into subcases depending on the complexity of the cut formula. Here we will assume that it is greater than 1. The proof then looks like the following.

$$\dfrac{\dfrac{\Gamma \succ \Delta, M, M'}{\Gamma \succ \Delta, M} \qquad M'', \Sigma \succ \Theta}{\Gamma, \Sigma \succ \Delta, \Theta}$$

Since M and M' are equivalent in $\mathscr{E}$, as are M and M'', it follows that M' and M'' are. We can then permute the cut upwards as follows.

$$\dfrac{\dfrac{\Gamma \succ \Delta, M, M' \qquad M'', \Sigma \succ \Theta}{\Gamma, \Sigma \succ \Delta, \Theta, M} \qquad M'', \Sigma \succ \Theta}{\Gamma, \Sigma, \Sigma \succ \Delta, \Theta, \Theta}$$

The new cuts can be eliminated by the induction hypothesis on the contraction count. The desired endsequent can then be obtained by repeated use of contraction and permutation rules.

The other cases involving [*W*L] and [*W*R] are similar.

The cases in which one or both cut formulas are parametric in their respective inferences are handled by the usual induction hypothesis on rank.

We will do a few cases in which both cut formulas are principle in their inferences.

Case: [$\wedge$]. Both cut formulas are principle, so the proof looks like the following.

$$\dfrac{\dfrac{\Gamma \succ \Delta, A \qquad \Gamma \succ \Delta, B}{\Gamma \succ \Delta, A \wedge B} \qquad \dfrac{A', B', \Sigma \succ \Theta}{A' \wedge B', \Sigma \succ \Theta}}{\Gamma, \Sigma \succ \Delta, \Theta}$$

In this proof, A and A', as well as B and B', are, respectively, equivalent in $\mathscr{E}$.

This is transformed into the following.

$$\dfrac{\Gamma \succ \Delta, B \qquad \dfrac{\Gamma \succ \Delta, A \qquad A', B', \Sigma \succ \Theta}{B', \Gamma, \Sigma \succ \Delta, \Theta}}{\Gamma, \Gamma, \Sigma \succ \Delta, \Delta, \Theta}$$

The new cuts can be eliminated using the induction hypothesis on complexity. The desired endsequent can then be obtained by repeated use of contraction and permutation rules.

The [$\neg\wedge$] and double negation cases are similar.

Case: [T]. The proof ends with the following.

$$\dfrac{\dfrac{\Gamma \succ \Delta, A}{\Gamma \succ \Delta, Tb} \quad \dfrac{A, \Sigma \succ \Theta}{Tc, \Sigma \succ \Theta}}{\Gamma, \Sigma \succ \Delta, \Theta}$$

In this proof, b and c are equivalent in $\mathscr{E}$.

This can be transformed into the following.

$$\dfrac{\Gamma \succ \Delta, A \quad A, \Sigma \succ \Theta}{\Gamma, \Sigma \succ \Delta, \Theta}$$

The new cut can be eliminated by appeal to the induction hypothesis on complexity, since the trace weight of the cut formula has been reduced.

Case: [$\neg$T]. The proof ends with the following.

$$\dfrac{\dfrac{\Gamma \succ \Delta, \neg A}{\Gamma \succ \Delta, \neg Tb} \quad \dfrac{\neg A, \Sigma \succ \Theta}{\neg Tc, \Sigma \succ \Theta}}{\Gamma, \Sigma \succ \Delta, \Theta}$$

In this proof, b and c are equivalent in $\mathscr{E}$.

This can be transformed into the following.

$$\dfrac{\Gamma \succ \Delta, \neg A \quad \neg A, \Sigma \succ \Theta}{\Gamma, \Sigma \succ \Delta, \Theta}$$

The new cut can be eliminated by appeal to the induction hypothesis on complexity, since the trace weight of the cut formula has been reduced.

Finally, we observe that in no case did the trace weight of a cut formula increase from the original cut to the new cuts in the transformed proof. □

□

Since this proof system is somewhat non-standard, we will demonstrate its adequacy by showing that it is equivalent to a fragment of the sequent system for Strong Kleene truth from Kremer (1988). Rather than give Kremer's rules, we will use the rules from $K3TL^{=}$, with restrictions that we will indicate. These rules are admissible in Kremer's system. The fragment in which we will be interested here is the quantifier-free fragment without identity axioms and whose identity rules are restricted to operate only on literals using the truth predicate. Further, the proofs are required to be *syntax consistent*, in the sense that for a given derivation, the equalities on the left and negations of equalities on the right of the end sequent do not imply, using classical equational logic, $\langle A\rangle = \langle B\rangle$, for any distinct formulas A and B.[8] Call this fragment KR. For a given $\mathscr{E}$ and an instance of a truth rule whose displayed premiss formula is $\pm A$ and whose conclusion is $\pm Tb$, say that a set of identities Ξ *underwrites* the application of the rule just in case Ξ contains the identity $\langle A\rangle = b$. Say that a set of identities underwrites a truth axiom of one of the followings forms,

[8] Syntax consistency says, roughly, that the set of equalities on the left and negations of equalities on the right can be extended to an annotation set.

$$\frac{\Gamma, \pm Ts \succ \Delta}{\Gamma, t = s, \pm Tt \succ \Delta}\ [{=}L1] \qquad \frac{\Gamma \succ \Delta, \pm Ts}{\Gamma, t = s, \succ \Delta, \pm Tt}\ [{=}L2]$$

$$\frac{\Gamma, \pm Ts \succ \Delta}{\Gamma, s = t, \pm Tt \succ \Delta}\ [{=}L3] \qquad \frac{\Gamma \succ \Delta, \pm Ts}{\Gamma, s = t, \succ \Delta, \pm Tt}\ [{=}L4]$$

$$\frac{\Gamma, \pm Ts \succ \Delta}{\Gamma, \pm Tt \succ \Delta, t \neq s}\ [{\neq}R1] \qquad \frac{\Gamma \succ \Delta, \pm Ts}{\Gamma \succ \Delta, \pm Tt, t \neq s}\ [{\neq}R2]$$

$$\frac{\Gamma, \pm Ts \succ \Delta}{\Gamma, \pm Tt \succ \Delta, s \neq t}\ [{\neq}R3] \qquad \frac{\Gamma \succ \Delta, \pm Ts}{\Gamma \succ \Delta, \pm Tt, s \neq t}\ [{\neq}R4]$$

$$\frac{\Gamma, A \succ \Delta}{\Gamma, T\langle A\rangle \succ \Delta}\ [TL] \qquad \frac{\Gamma \succ \Delta, A}{\Gamma \succ \Delta, T\langle A\rangle}\ [TR]$$

$$\frac{\Gamma, \neg A \succ \Delta}{\Gamma, \neg T\langle A\rangle \succ \Delta}\ [\neg TL] \qquad \frac{\Gamma \succ \Delta, \neg A}{\Gamma \succ \Delta, \neg T\langle A\rangle}\ [\neg TR]$$

Figure 3: Identity and truth rules in KR

- $\Gamma, \pm Tb, \Sigma \succ \Delta, \pm Tc, \Theta$, or
- $\Gamma, Tb, \neg Tc, \Sigma \succ \Delta$,

just in case Ξ contains $b = c$. A set of identities Ξ underwrites a proof just in case Ξ underwrites each truth rule and truth axiom in the proof.

The axioms in KR can be used for arbitrary formulas, rather than being restricted to atoms. This is not a problem, since K3TL$\mathcal{E}$ allows one to prove that the axioms hold for arbitrary formulas.

Lemma 7. *The axioms* [Id] *and* [K3] *are derivable with an arbitrary formula* A *replacing* p. *The axioms* [Cl] *are derivable with an arbitrary* T*-free formula* B *replacing* p.

The proof is by induction on the construction of A, or B. The proof is routine, so we omit it.

With that lemma in hand, we can now state the equivalence between the systems.

Theorem 8.

1. *Let* Π *be a* K3TL$\mathcal{E}$ *derivation of* $\Gamma \succ \Delta$. *Then there is a derivation of* $\Gamma, \Xi \succ \Delta$ *in* KR*, where* Ξ *is a set of identities that underwrites each truth rule and truth axiom used in* Π.
2. *Let* Π *be a proof of* $\Gamma, \Sigma \vdash \Delta, \Theta$ *in* KR*, where* Σ *is a set of identities introduced via identity rules and* Θ *is a set of negated identities introduced via identity rules. Then* $\Gamma \vdash \Theta$ *is derivable in* K3TL$\mathcal{E}$*, provided* $\mathcal{E}$ *contains* $\Sigma \cup \Theta^*$*, where* $\Theta^* = \{s = t : s \neq t \in \Theta\}$.

For reasons of space, we will sketch the proof. For 1, one translates a proof Π of $\Gamma \succ \Delta$ in K3TL$\mathcal{E}$ into a proof Π' in KR. We briefly describe some of the cases. Whenever a truth rule is used in Π, a truth rule is used in Π', followed by an appropriate identity rule if the principal formula in the rule in Π was of the form $\pm Ta$ rather than $\pm T\langle A\rangle$. These additional identities make up Ξ.

For 2, one translates a proof Π' of $\Gamma, \Sigma \succ \Delta, \Theta$ into a proof Π of $\Gamma \succ \Delta$ in $K3TL\mathcal{E}$. The important cases for this proof are the identity cases. These are taken care of by global transformations on trace trees in Π.

We will note that the theorem, and proof, carry over to LP, using Kremer's sequent calculus for LP.

The restriction in KR to use identity rules only on literals using the truth predicate is to facilitate the proof of theorem 8, and it is not a great restriction. Nothing additional is provable if the identity rules are allowed to substitute into truth atoms in complex formulas.

Proposition 9. *Let* A *be a formula with at least one occurrence of* Ta *and let* A' *be* A *with one or more occurrences of* Ta *replaced by* Tb*. The rules*

$$\frac{\Gamma, A \succ \Delta}{a = b, \Gamma, A' \succ \Delta}\,[=L] \qquad \frac{\Gamma, A \succ \Delta}{b = a, \Gamma, A' \succ \Delta}\,[=L]$$

$$\frac{\Gamma, A \succ \Delta}{\Gamma, A' \succ \Delta, a \neq b}\,[\neq R] \qquad \frac{\Gamma, A \succ \Delta}{\Gamma, A' \succ \Delta, b \neq a}\,[\neq R]$$

are admissible in KR*.*

The proof is by induction on the construction of A. It is straightforward, so we omit it. We will turn to the conclusion.

4 Conclusion

We began with the aim of providing some proof-theoretic explanation of why the fixed-point theories of truth based on K3 and LP are non-trivial. Focusing on K3, we proved that a very basic system, one with no additional syntactic theory, is non-trivial by a balancing argument. The argument extends to LP. Enriching this system with a modest syntactic theory and identity, leads to difficulties showing that cut is eliminable. Indeed, the enrichment requires a more complicated definition of triviality. We proposed non-standard systems that internalize the syntactic theory in the annotation sets and their effect on the truth rules. A cut elimination argument can be carried out for those systems, showing that the systems are non-trivial in the original sense that $\emptyset \succ \emptyset$ is not derivable. Finally, we showed that the systems are intertranslatable with fragments of Kremer's system.

There is still work to be done. It would be good to obtain a completeness result of some kind for the systems with annotation sets. We also hope to generalize the initial balancing argument to work more broadly. But, we have made some progress towards our initial goal of getting a proof-theoretic justification for the non-triviality of naive truth in K3 and LP.

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