Monash University

Department of Philosophy and Bioethics

PhD Thesis

Translational Embeddings in Modal Logic

Rohan French
BA Hons., BCSc

(July, 2010)

Contents

Summary

Often when we do philosophy we want to define less well understood notions with which we are concerned in terms of those which we have a much firmer grasp on. Hence, to take a classical example, we often try to define knowledge in terms of belief (and some auxiliary notions such as truth, justification and so on). Such definitions give rise to translations, functions which map the terms to be explained to their intended (simpler) explanation, and it is these which are our objects of study.

In particular we will be focused on providing a systematic study of the formal uses of translations between logics, and what they can tell us about the logics involved. Our focus will be on propositional modal logics, both for reasons of explanatory simplicity, and also as modal logic remains one of the best formal tools we have for explicating a wide variety of live philosophical problems.

Statement of Originality

I declare that this thesis contains no material which has previously been submitted for a degree or diploma at any university and, to the best of my knowledge and belief, this thesis contains no material which has previously been published or written by another person, except when due reference is made in the text of the thesis.

Rohan French
December 13, 2010

Acknowledgments

I would like to begin by thanking Lloyd Humberstone, whose enthusiasm for interesting problems and ideas made this thesis possible. I would also like to thank Sam Butchart, Graham Oppy and Greg Restall, conversations with whom helped me clarify much of my thinking concerning a variety of issues regarding translations between logics. Thanks are also due to the rest of the Monash philosophy department, whose support and resources I should have made fuller use of.

I was also lucky to be part of an email conversation which occurred between Lloyd Humberstone, Allen Hazen, F.J. Pelletier, and A. Urquhart concerning an error in Pelletier & Urquhart [2003], which was the impetus for the follow up paper Pelletier & Urquhart [2008]. The example in Chapter 5.2.2 owes a heavy debt to a similar example presented there by Allen Hazen.

This thesis would not have been possible without the love and support of my parents and Laura Birch, who I'd like to thank for listening to me even if they didn't understand what I was talking about.

The research herein was conducted under the financial assistance of a Australasian Postgraduate Association Scholarship, for which the author

was extremely grateful.

I

Introduction

In this thesis we will be considering translations between logics – these being functions which map formulas in the language of one logic (the source logic) to formulas in the language of the other (the target logic). Translations are a very useful tool for studying the interrelationships between different logics, as well as for the proper explication, and occasionally solution, of some philosophical problems. Consider the following example from the philosophy of mathematics. If mathematical intuitionism were correct one might have suspected that 'illegitimate' use of the principle of excluded middle, and other classically valid but intuitionistically invalid principles of reasoning, would have rendered some of our mathematical results incorrect, if not inconsistent – in the same way what treating principles such as affirming the consequent as valid principles of reasoning can lead to inconsistency. But it appears as if the use of the principle of excluded middle in mathematics has not led us astray, and so the mathematical intuitionist might be led to wonder what exactly is going on here. In the 1920's Andrei Kolmogorov addressed this problem, showing why the use of the principle of excluded middle in mathematical reasoning "has not yet led to contradictions and also why the very illegitimacy has often gone unnoticed" [van Heijenoort 1967, p.416]. To do this what Kolmogorov showed was that we could translate classical logic into intuitionistic logic, and thus any classical reasoning can be transformed – using the translation in question – into reasoning which is intuitionistically acceptable. We will not address here how successful Kolmorogov's response to this problem is, merely mentioning it as an example of where translations can be used (at least potentially) to solve philosophical problems.

In this thesis we will be concerned with giving a systematic study of the formal uses of translations and what they can tell us about different logics. In particular we will be concerned with translations between modal logics, both for reasons of explanatory simplicity and also because, in the author's opinion, modal logic remains one of the best formal tools we have for properly explicating a wide variety of live philosophical problems.

In Chapter 2 we will give a taxonomy of translations based on a series of properties of translations which have appeared in other studies of translations – notably Prawitz & Malmnäs [1968] and Wójcicki [1988]. The idea behind our taxonomy is to try to properly characterize the variety of translations which appear in the literature, focusing in particular on properties of how the translation acts on formulas, before zooming in on the particular kind of translations with which we will be predominantly concerned with in the rest of the thesis.

Gödel showed that we can translate Intuitionistic Logic into the normal modal logic **S4**. It was noticed in the 60's by A. Grzegorczyk that Intuitionistic Logic can also be translated into a proper extension **S4** using the same translation. The question then arises of which logics we can translate Intuitionistic Logic into using this translation. In Chapter 3 we investigate the general phenomenon which this is an instance of, investigating the notion of the *range* of a translation. The idea here is to investigate the structure of the set of logics which a given source logic can be faithfully embedded into by a given translation. Our focus here will be algebraic in nature, focusing on minimal and maximal such elements, and the structures which such set of logics form under the $\subseteq$ relation. In Chapter 4 we will then focus on some particular examples of this phenomenon, ending with a conjecture concerning a simple and important translation.

In Chapter 5 we consider a notion which has appeared in a number of places called translational equivalence, focusing in particular on some recent results concerning translational equivalence due to F.J. Pelletier and A. Urquhart (Pelletier [1984], Pelletier & Urquhart [2003], Pelletier & Urquhart [2008]). This chapter closes by considering the view often posited in the literature that translational equivalence captures our notion of equivalence between logics.

In Chapter 6 we turn to considerations of non-normal modal logics, and the translations between them. Here our focus will be on translations which embed weak non-normal modal logics into the smallest normal modal logic **K**. In particular we consider some questions concerning a rather novel translation proposed by M. Brown in a paper on the logic of action. This naturally raises the question of whether there is a translation of a particular sort which translates the smallest congruential modal logic **E** into the smallest normal modal logic **K** – a question which we then go on to answer positively.

Before going on, though, we will need to go through some logical preliminaries.

1.1 Logical Preliminaries

Propositional logics will largely be at issue here, with predicate logic referred to only in digressions or to make the occasional formal point more clearly. We will take our propositional languages to be built from a stock of denumerably many propositional variables $p_0, p_1, p_2, \ldots$ using some stock of primitive connectives in the usual way. We will often use the abbreviations p, q, r to stand for the first three propositional variables. Given a formula A, let $comp(A)$ be the number of connectives present in A – i.e. $comp(p_i) = 0$, $comp(\#(A_1, \ldots, A_n)) = [\Sigma_{i=0}^{n} comp(A_i)] + 1$.

Throughout this thesis we will be concerned with two different conceptions of logic, involving different choices of what syntactic entities our logic will be concerned with – what syntactic entities we are taking to be provable or unprovable in our logic. To make this issue clearer we will introduce the idea of a sequent. Letting Γ, Δ be finite sets of formulas of the language under consideration, the pair $\langle \Gamma, \Delta \rangle$ is a *sequent* over that language

- for the logical framework SET-FMLA if $\Delta = \{B\}$ for some formula B
- for the logical framework FMLA if $\Delta = \{B\}$ for some formula B and $\Gamma = \emptyset$.

Predominately we will be concerned here with the conception of logics as sets of formulas (the FMLA framework) although we will occasionally make use of the greater generality of thinking about some issues in the SET-FMLA framework. In particular we will be interested occasionally in consequence relations, which places some restrictions on what kinds of sequents are provable in our logic. A relation $\vdash : \wp(L) \times L$ over a language L is a *consequence relation* if it is closed under the following three rules (R), (M) and (T) for all formulas A, B and all sets of formulas Γ and Γ'.

- (R): $A \vdash A$
- (M): If $\Gamma \vdash B$ then $\Gamma, A \vdash B$.
- (T): If $\Gamma, A \vdash B$ and $\Gamma' \vdash A$ then $\Gamma, \Gamma' \vdash B$.

Here we are writing $\Gamma \vdash A$ to mean $\langle \Gamma, A \rangle \in \vdash$ – taking the usual abbreviation of writing $\Gamma, A \vdash B$ instead of $\Gamma \cup \{A\} \vdash B$. A consequence relation $\vdash$ will be said to be *substitution-invariant* if the set of sequents which comprise it are closed under uniform substitution (of arbitrary formulas for propositional variables). A logic in the FMLA framework will be taken to be any non-empty set of formulas S closed under uniform substitution. Such a logic will be said to be consistent whenever it does not contain all formulas in the language. We will equally often shift between writing $A \in \mathsf{S}$ and $\vdash_{\mathsf{S}} A$ when we are talking about formulas being theorems of logics in the FMLA framework.

In what follows we shall predominantly be concerned with modal logics – which extend the language of classical logic by the addition of a non-boolean primitive operator $\Box$. More formally we will take the modal language to be constructed out a denumerable set of propositional variables, as above, using the connectives $\{\neg, \rightarrow, \Box\}$ for the sake of definiteness – although we will casually refer to the other (in this setting derived) classical connectives as well as the truth and falsity constants $\top$ and $\bot$ in our exposition. A set of formulas from the modal language S is a modal logic (in the FMLA framework) whenever it fulfils the following two conditions.

- S contains all the classical tautologies
- S is closed under the rules of Modus Ponens and Uniform Substitution.

The smallest modal logic we will occasionally refer to as **L**.

Following the nomenclature of Chellas [1980] we will say that a modal logic is *congruential* whenever it is also closed under the following rule.

$$\textbf{RE}: \quad \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$$

The least such modal logic we will refer to as **E**. One useful fact about modal logics which are congruential is that whenever $A \leftrightarrow B$ is provable in such a logic then we can replace occurrences of A in a formula with occurrences of B without affecting the provability of the formula in question.

A modal logic will be said to be *monotone* if it is closed under the following rule.

$$\textbf{RM}: \quad \frac{A \to B}{\Box A \to \Box B}$$

The least such modal logic we will refer to as **EM** – as we could also equally axiomatize this logic as the least congruential modal logic which contains the following axiom schemata.

$$\textbf{M}: \quad \Box(A \wedge B) \to (\Box A \wedge \Box B).$$

We will say that a modal logic is *regular* if it is closed under the following rule.

$$\textbf{RR}: \quad \frac{(A \wedge B) \to C}{(\Box A \wedge \Box B) \to \Box C}$$

The least such modal logic we will refer to as **EMC** – as again we could axiomatize this logic as the least congruential modal logic which contains the axiom schema **M** as well as the following:

$$\textbf{C}: \quad (\Box A \wedge \Box B) \to \Box(A \wedge B).$$

Lastly, and most importantly we can call a modal logic *normal* if it is regular and contains the formula **N** ($= \Box\top$). More conveniently though we will say that a modal logic S is normal iff it contains the axiom **K**

$$\textbf{K}: \quad \Box(A \to B) \to (\Box A \to \Box B).$$

and is closed under the rule of necessitation.

$$\textbf{RN}: \quad \frac{A}{\Box A}$$

Following terminology laid down in Segerberg [1971a], let us say than a modal logic S extending the smallest normal modal logic **K** is *quasi-normal*. In particular this means that all of the normal modal logics are also quasi-normal.

We will make use of the above rules when giving our infrequent proofs in modal logics – in particular making use of the fact that normal modal logics are closed under the rules **RM** and **RR**. We will refer to the corresponding versions of the above rules using $\Diamond$ instead of $\Box$ as $\Diamond$-**RE**,$\Diamond$-**RM**, $\Diamond$-**RR**. It is relatively easy to derive the $\Diamond$-versions of **RE** and **RM** from the corresponding $\Box$-versions of these rules. For example:

(1)	$A \rightarrow B$	*Hypothesis*
(2)	$\neg B \rightarrow \neg A$	(1), TF
(3)	$\Box\neg B \rightarrow \Box\neg A$	(2), **RM**
(4)	$\neg\Box\neg A \rightarrow \neg\Box\neg B$	(3), TF
(5)	$\Diamond A \rightarrow \Diamond B$	(4), $Def-\Diamond$.

In the above proof we made use on lines (2) and (4) of a rule (TF). This is the following rule of proof where B follows from $A_1,\ldots,A_n$ using Truth Functional reasoning.

$$(TF) \qquad \frac{A_1,\ldots,A_n}{B}$$

This rule is slightly odd in not having a fixed number of premises, as well as there being no application of the rule of which all other applications are substitution instances. Nonetheless, we will find it very convenient to use – shortening the lengths of proofs substantially and removing overly involved classical manoeuvring.

Given this rule (TF) we can give an alternative characterization of a modal logic as a set of formulas closed under uniform substitution and the rule (TF) – the classical tautologies being instances of the rule where $n = 0$, and closure under Modus Ponens following from instances of the rule where $n = 2$, $A_1 = B_1 \rightarrow B$ and $A_2 = B_1$.

The smallest normal modal logic extending a normal modal logic S in which all substitution instances of the formula A are provable we will denote by $\mathsf{S} \oplus \{A\}$. We will often suppress the use of "{" and "}", referring to this logic as $\mathsf{S} \oplus A$. We will often denote the system $\mathbf{K}\oplus\{\mathbf{X},\mathbf{Y}\}$ by **KXY** when **X** and **Y** are labels for modal axioms. We will denote the smallest modal

Label	Axiom
T	$\Box p \to p$
D	$\Box p \to \Diamond p$
4	$\Box p \to \Box\Box p$
5	$\Diamond p \to \Box\Diamond p$
B	$p \to \Box\Diamond p$
U	$\Box(\Box p \to p)$
Ver	$\Box\bot$
G (.2)	$\Diamond\Box p \to \Box\Diamond p$
Grz	$\Box(\Box(p \to \Box p) \to p) \to p$
GL (**W**)	$\Box(\Box p \to p) \to \Box p$
.4 (**ww5**)	$p \to (\Diamond\Box p \to \Box p)$
w4	$\Box p \wedge p \to \Box\Box p$
Alt$_n$	$\Box p_0 \vee \Box(p_0 \to p_1) \vee \ldots \vee \Box(p_0 \wedge \ldots \wedge p_{n-1} \to p_n)$

Figure 1.1: A list of the common modal axioms we will be referring to. The names in parenthesis being other common names for the axiom in question which can be found in the literature.

logic extending S which proves all instances of an axiom A by $\mathsf{S}+A$. Following Chellas [1980], if $\mathbf{X}$ is a label for an axiom of the form $A \to B$ then $\mathbf{X}_c$ will denote the axiom $B \to A$, and $\mathbf{X!}$ the axiom $A \leftrightarrow B$. So, for example, the axiom $\mathbf{D}_c$ is the formula $\Diamond p \to \Box p$, and $\mathbf{T!}$ is the axiom $\Box p \leftrightarrow p$ (to mention two examples we will encounter later, $\mathbf{D}$ and $\mathbf{T}$ being in Figure 1.1). We will follow the standard convention of referring to the logics **KT4** and **KT5** (or equivalently **KT45** or **KTB4**) as **S4** and **S5** respectively.

Throughout this thesis there are a great many proofs which proceed by induction upon the length of derivations of a formula A in a normal modal logic **KXY**. For such proofs, unless otherwise specified, we will be thinking of them as being axiomatized using all the classical rules for the primitive classical connectives (i.e. all the rules for $\{\to, \neg\}$), as well as the axioms **K**, **X**, and **Y** and our rules to be Modus Ponens, uniform substitution and the rule of necessitation.

In what follows we will make quite much of use of the very elegant model theory for normal modal logics. A *Kripke model* $\mathcal{M}$ is an ordered triple $\langle W, R, V\rangle$ where $W \neq \emptyset$ is a set of points, R a binary relation on W ($R \subseteq W \times W$) and V a function which maps the propositional variables to subsets of W – the points at which the variable in question is true. The structure $\langle W, R\rangle$ we will refer to as a *frame* – with a model $\langle W, R, V\rangle$ being a *model on the frame* $\langle W, R\rangle$. Letting R^0xy be $x = y$, we can define $R^{n+1}xy$ as $\exists z(R^n xz \wedge Rzy)$. Given an accessibility relation R we can define the *ancestral* of the accessibility relation R^* as $\{\langle x, y\rangle : \exists n \in Nat, R^n xy\}$ – this being the reflexive, transitive closure of R.

We can define truth of a formula A at a point $x \in W$ in a model $\mathcal{M} = \langle W, R, V\rangle$ (written as $\mathcal{M} \models_x A$) inductively as follows.

$$
\begin{array}{rcl}
\mathcal{M} \models_x p_i & \text{if and only if} & x \in V(p_i). \\
\mathcal{M} \models_x A \to B & \text{if and only if} & \mathcal{M} \not\models_x A \text{ or } \mathcal{M} \models_x B. \\
\mathcal{M} \models_x \neg A & \text{if and only if} & \mathcal{M} \not\models_x A. \\
\mathcal{M} \models_x \Box A & \text{if and only if} & (\forall y)(Rxy \Rightarrow \mathcal{M} \models_y A).
\end{array}
$$

We will say that a formula A is *true throughout a model* ($\mathcal{M} \models A$) whenever for all $x \in W$, $\mathcal{M} \models_x A$. A formula A will be said to be *valid on a frame* $\mathfrak{F} = \langle W, R \rangle$ ($\mathfrak{F} \models A$) whenever it is true throughout all models on that frame. If $\mathcal{C}$ is a class of frames then we will say that A is valid over $\mathcal{C}$ whenever A is valid on all frames $\mathfrak{F} \in \mathcal{C}$ – notating this as $\mathcal{C} \models A$. A modal logic S is *sound* w.r.t. a class of frames $\mathcal{C}$ whenever $A \in \mathsf{S}$ implies that $\mathcal{C} \models A$, and *complete* w.r.t. a class of frames whenever $\mathcal{C} \models A$ implies that $A \in \mathsf{S}$. Sometimes when S is both sound and complete w.r.t. a class of frames $\mathcal{C}$ we may say that S is *determined* by $\mathcal{C}$. These notions generalize to classes of models in the obvious way.

Given a model $\mathcal{M} = \langle W, R, V \rangle$ and a point $x \in W$ define the following model $\mathcal{M}^x = \langle W^x, R^x, V^x \rangle$ where:

- $W^x := \{y | R^* xy\}$
- $R^x := R \cap W^x \times W^x$
- $V^x(p_i) := V(p_i) \cap W^x$

We will call a model $\mathcal{M}^x$ a *point-generated submodel of* $\mathcal{M}$, picking out the point-generated submodel where x is the generating point as *the submodel of* $\mathcal{M}$ *generated by* x. The usefulness of generated submodels can be seen from the following well known result.

Theorem 1.1.1 (Generation Theorem). *For all models $\mathcal{M} = \langle W, R, V \rangle$ we have the following for all points $x \in W$ and all formulas A.*

$$\mathcal{M} \models_x A \text{ if and only if } \mathcal{M}^x \models_x A.$$

We will also make tacit use of standard soundness and completeness results for modal logics with respect to certain classes of frames as given in Chellas [1980]. For example the normal modal logic **KT** is sound and complete w.r.t. the class of frames $\mathcal{C}_{\mathbf{KT}}$ which satisfy the first order condition $(\forall x)(Rxx)$ – i.e. the reflexive frames. The table below gives a list of some common (and not so common) modal logics we will be concerned with, and the first order condition on frames which they are sound and complete with respect to.

Label	First order condition on Frames.	
T	$(\forall x)(Rxx)$	[Reflexive]
D	$(\forall x)(\exists y)(Rxy)$	[Serial]
4	$(\forall x)(\forall y)(\forall z)(Rxy \& Ryz \rightarrow Rxz)$	[Transitive]
5	$(\forall x)(\forall y)(\forall z)(Rxy \& Rxz \rightarrow Ryz)$	[Euclidean]
B	$(\forall x)(\forall y)(Rxy \rightarrow Ryx)$	[Symmetric]

II

A Taxonomy of Translations

The main objects of study of this thesis are translations between logics, functions which map formulas of one formal language (our source language) to formulas of another (our target language) which possess certain properties which make them more deserving of being called *translations* than arbitrary functions from the source language to the target language. There is a certain degree of freedom as to what kind of functions translations should be, and what kinds of properties they should possess, and it is these issues which we will address here. In particular we will begin by sketching out a taxonomy of different kinds of translations, before then going on to give some more information on a particular kind of translation with which we will primarily be interested, the modal-to-modal translation. We will then close the chapter by considering some examples which illustrate the usefulness of such translation to properly stating and addressing certain kinds of philosophical problems.

Throughout we will predominantly be concerned with logics thought of as sets of formulas, although at times we will find it more convenient to state things in terms of consequence relations – in which case the relevant point concerning logics as set of formulas will follow by considering the consequences of the empty set. Having made this point, let $\vdash_0$ and $\vdash_1$ be consequence relations on propositional languages L_0 and L_1, $\vdash_0$ being the *source* logic, and $\vdash_1$ the *target* logic of the translation. Then a function $\tau\colon L_0 \to L_1$ is a *translation*. Moreover, we will say that τ *embeds* $\vdash_0$ *into* $\vdash_1$ whenever, for all formulas $A_1,\ldots,A_n,B$ we have that:

$$A_1,\ldots,A_n \vdash_0 B \text{ only if } \tau(A_1),\ldots,\tau(A_n) \vdash_1 \tau(B). \tag{2.1}$$

Translations which fulfil this rather weak condition have received relatively little attention in the literature – notable exceptions being Silva *et al*. [1999] and Humberstone [2000]. In Silva *et al*. [1999] translations between consequence operations which fulfil condition (2.1) are investigated with respect to their category-theoretic properties, where it is shown that logics and the translations between them form a category. In Humberstone [2000] these translations are used to formally define what it is for a logic to deviate from classical logic[1] by endorsing classically invalid principles. The idea here is that a logic $\vdash_0$ is *contra-classical* if there is no translation τ[2] for which the analogue of (2.1) holds with $\vdash_1$ being the classical consequence relation $\vdash_{CL}$. The idea here being that contra-classical logics deviate from classical logic by commission in a non-superficial manner – the presence of a translation τ showing the contra-classicality to be merely superficial, a good example of this superficial contra-classicality being the $\vee$-treated-like-$\wedge$ logic mentioned in Humberstone [2000, p.440f.].

Usually though, we will be concerned with translations which at minimum fulfil the following condition for all formulas $A_1,\ldots,A_n,B$.

[1] The idea of 'contra-classicality' can quite easily be extended to that of a logic being 'contra-S' for any logic S, later sections of Humberstone [2000] being devoted to contra-intuitionistic logics for example.

[2] Which also fulfils the conditions which we will be calling variable-fixedness and compositionality

$$A_1, \ldots, A_n \vdash_0 B \text{ if and only if } \tau(A_1), \ldots, \tau(A_n) \vdash_1 \tau(B). \tag{2.2}$$

We will say that a translation τ *faithfully embeds* $\vdash_0$ *into* $\vdash_1$ whenever (2.2) is fulfilled, the word *faithful* here recording the 'if' direction of the above claim. Translations like this are called *conservative* in Feitosa & D'Ottaviano [2001] and Silva *et al*. [1999], *unprovability-preserving* in Inoué [1990] and *exact* in Pelletier & Urquhart [2003]. We will largely be interested in logics in the FMLA framework, and thus will be concerned predominantly with the $n = 0$ cases of (2.1) and (2.2).

If this was all there was to say about translations then things would be quite dull, as we could faithfully embed all of the logics which are are interested in into each other using a very uninteresting translation.

Let S and S' be two consistent logics in the FMLA framework such that S and S' are non-empty. Let $B_\top$ be an arbitrary theorem of our target logic S' (i.e. $B_\top \in \mathsf{S}'$), and $B_\perp$ be an arbitrary formula such that $B_\perp \notin \mathsf{S}'$. Then we can define $\tau_{(\mathsf{S},\mathsf{S}')}$ as follows:

$$\tau_{(\mathsf{S},\mathsf{S}')}(A) = \begin{cases} B_\top, & A \in \mathsf{S} \\ B_\perp, & A \notin \mathsf{S}. \end{cases}$$

Theorem 2.0.2. *For all formulas A, and all consistent, non-empty* FMLA *logics S and S' we have the following.*

$$A \in \mathsf{S} \textit{ if and only if } \tau_{(\mathsf{S},\mathsf{S}')}(A) \in \mathsf{S}'.$$

Proof. For the 'only if' direction suppose that $A \in \mathsf{S}$. Then $\tau_{(\mathsf{S},\mathsf{S}')}(A) = B_\top$, and thus by definition $\tau_{(\mathsf{S},\mathsf{S}')}(A) \in \mathsf{S}'$. For the 'if' direction suppose that $A \notin \mathsf{S}$. Then $\tau_{(\mathsf{S},\mathsf{S}')}(A) = B_\perp$, and thus by definition $\tau_{(\mathsf{S},\mathsf{S}')}(A) \notin \mathsf{S}'$ as desired. □

The existence of such a translation tells us nothing about the logics involved at all. In the light of such results then, in order for the enterprise to be interesting we will want to look at certain ways of restricting translations so that they fulfil certain conditions. In what follows we will be concerned with four main features of the way in which translations act on formulas, which will allow us to map out the whole translational landscape. First, though, we will need the following definition. Let us say that a set T of translations from L_0 to L_1 is *recursively interdependent* if, for every $\tau \in T$ and all primitive n-ary connectives $\# \in L_0$ there are $m_1, \ldots, m_n \in Nat$ and translations $\tau'_{1,1}(A_1), \ldots, \tau'_{1,m_1}(A_1), \ldots, \tau'_{n,1}(A_n), \ldots, \tau'_{n,m_n}$ in T such that:

$$\tau(\#(A_1, \ldots, A_n)) = \#^\tau(\tau'_{1,1}(A_1), \ldots, \tau'_{1,m_1}(A_1), \ldots, \tau'_{n,1}(A_n), \ldots, \tau'_{n,m_n}(A_n))$$

Where $p_1, ..., p_{m_1}, p_{m_1+1}, \ldots, p_{m_1+...+m_{n-1}+1}, \ldots, p_{m_1+...+m_n}$ are all and only the propositional variables contained in $\#^\tau$. Note that if T is recursively interdependent then every translation $\tau \in T$ is such that, for all formulas $A \in L_0$ there is a unique formula $A^\tau \in L_1$ such that $\tau(A) = A^\tau$.[3]

- Variable-Fixed: A translation τ is *variable-fixed* whenever for all propositional variables p_i we have that $\tau(p_i) = p_i$.
- Schematic: A translation τ is *schematic* whenever there is a formula $C(p_1)$ such that $\tau(p_i) = C(p_i)$.
- Compositional: A translation τ is *compositional* whenever for every primitive n-ary connective $\#(A_1, \ldots, A_n)$ in the language of L_0 (the domain of τ) there is a formula $\#^\tau(p_1, \ldots, p_n)$ in L_1 (the codomain of τ) constructed out of exactly the propositional variables $p_1, \ldots, p_n$ such that $\tau(\#(A_1, \ldots, A_n) = \#^\tau(\tau(A_1), \ldots, \tau(A_n))$.
- Recursive: A translation τ is *recursive* if it is a member of a recursively interdependent set of translations.

Translations which are both variable fixed and compositional will, following Tokarz & Wójcicki [1971] and Wójcicki [1988], be called *definitional*. Definitional translations are so named because the presence of one

[3]This can be shown by induction upon the complexity of formulas in L_0. This way of formulating the recursiveness condition upon translations is due to S. Kuhn.

between two logics indicates that we can define the connectives of the source logic in terms of those of the target logic – the definitions of each of the primitive connectives of the source logic simply being the relevant translation clause with all references to τ removed. It is worth noting that all definitional translations are also schematic – the formula $C(p_1)$ there being simply p_1 itself.

It is worth stressing the difference between a translation τ being compositional and it being recursive. With a compositional translation of a subformula A is determined by the primary connective of A. This need not be so for a recursive translation – good examples of this being the non-compositional but recursive translations for which $\tau(A)$ is 'polarity dependent' (cf. the translations in §2.0.3 and §2.0.4). A recursive translation is one where the result of translating a formula depends on the structure of the whole formula, rather than just a part of it, while a compositional translation is one where the result of translating a formula is the result of translating all its sub-formulas. We can see how this makes all compositional translations recursive – as the set $\{\tau\}$ is recursively interdependent for a compositional translation.

We will occasionally want to consider translations which do not 'interfere' with any but a select subclass of the connectives of a language – the main example be the modal-to-modal translations considered at the end of this chapter. To this end say that a translations τ is *homonymous* on a connective # whenever $\tau(\#(A_1,\ldots,A_n)) = \#(\tau(A_1),\ldots,\tau(A_n))$, and homonymous on a set of connectives when it is homonymous on every connective in that set.

These four properties of translations allow us to begin to sketch out a taxonomy of translations (Figure 2). This taxonomy (which we will further refine in section 2.0.5) leaves us with nine different types of translations – T_1 to T_9 – of which only T_1 (the definitional translations) and T_9 (the 'one-off' translations) have names. What we will now do is give examples of translations of types T_1 through to T_6 before, in section 2.0.5,

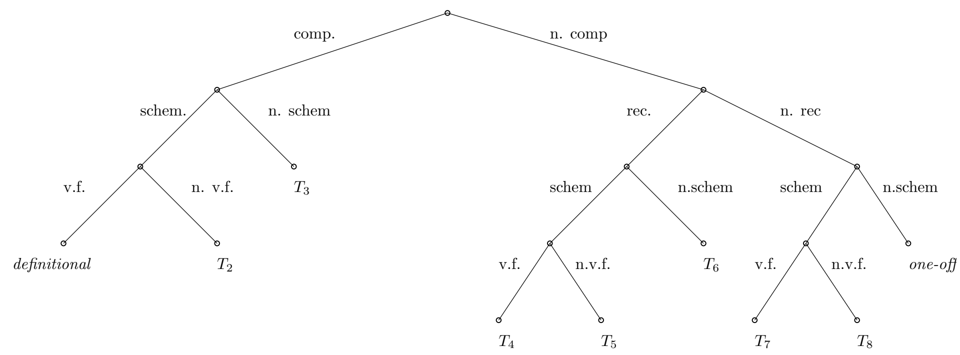

Figure 2.1: A preliminary Taxonomy of Translations.

refining how we classify the non-recursive translations in order to arrive at our final taxonomy. It is worth noting that our approach to classifying translations follows more closely those of Wójcicki [1988] and Prawitz & Malmnäs [1968] in focusing on the ways in which the translation acts on formulas, than the approach to classifying translations in terms of the way in which they relate logics – as is done in Silva *et al.* [1999] and Feitosa & D'Ottaviano [2001]. This mostly has to do with what our objects of investigation are here – we are concerned with what can be said about translations themselves, as opposed to translations in conjunction with source and target logics.

One thing which the present author found somewhat interesting was the lack of any T_3-translations in the literature, especially considering the fact that all the other translations can be found being used out in the logical wild as it were. One can quite easily concoct translations which are compositional while failing to be schematic, but it is harder to find any interesting use for such translations.[4] As such we will ignore them in what

[4]One can of course construct a T_3 translation out of any definitional or T_2 translation. In the definitional case all one needs to do is let $\tau(p_i) = p_{i+1} \wedge p_0$, moreover if the target logic of the original translation is substitution invariant then this new T_3-translation will also faithfully embed the source logic of the original definitional translation into its target logic.

follows.

2.0.1 T_1 – Embedding Classical Logic into Łukasiewicz's Three-Valued Logic

Recall that Łukasiewicz's three-valued Logic Ł$_3$ is the set of formulas which receive the value 1 under all ternary valuations v which assign either 1,2 or 3 to each propositional variable p_i, where $v(A)$ is calculated according to the tables given in Figure 2.0.1.

$\to$	1	2	3
$*1$	1	2	3
2	1	1	2
3	1	1	1

$\neg$	
$*1$	3
2	2
3	1

Figure 2.2: Łukasiewicz's Ł$_3$

Consider the following translation τ taken from Tokarz & Wójcicki [1971, p.126], which faithfully embeds Łukaseiwicz's 3-valued logic Ł$_3$ into Classical propositional logic (CL) with $\{\to, \neg\}$ as primitive.

$$\tau(p_i) = p_i;\ \tau(A \to B) = \tau(A) \to (\tau(A) \to \tau(B));\ \tau(\neg A) = \tau(A) \to \neg\tau(A).$$

As we can see this translation is both variable fixed and compositional, and hence is what we are calling a definitional translation, a translation of type T_1. That it is a definitional translation which faithfully embeds CL into Ł$_3$ is encapsulated in Theorem 2.0.5. Before going on to prove this we will require a Lemma regarding valuations.

Definition 2.0.3. Given a ternary assignment of values to the propositional variables v define a boolean assignment of values to the propositional variables v' as follows.

$$v'(p_i) = \begin{cases} T, & v(p_i) = 1 \\ F, & v(p_i) \neq 1. \end{cases}$$

Lemma 2.0.4. *Let v_3 be a ternary assignment of values to the propositional variables, v'_3 be the corresponding boolean assignment. Let v and v' be the corresponding unique ternary and boolean valuations. Then for all formulas A we have that $v'(A) = T \iff v(\tau(A)) = 1$.*

Proof. By induction upon the construction of A, the only case of interest being in the inductive step where $A = \neg B$ and $A = B \to C$. Note that we will occasionally, for the sake of convenience, write $v(A \to B) = i/j \to k/l$ where $i, j, k, l \in \{1, 2, 3\}$ to mean that either $v(A) = i \to k$ or $v(A) = j \to l$.

For the case where $A = \neg B$, suppose that $v'(A) = T$. Then $v'(B) = F$ and we need then to show that $v(\tau(\neg B)) = 1$, given that by the Inductive Hypothesis we know that $v(\tau(B)) \neq 1$.

$$\begin{aligned} v(\tau(\neg B)) &= v(\tau(B)) \to \neg v(\tau(B)) \\ &= 2/3 \to \neg(2/3) \\ &= 2/3 \to 2/1 \\ &= 1 \end{aligned}$$

Suppose then that $v'(A) = F$. Then $v'(B) = T$ and we need to show that $v(\tau(\neg(B)) \neq 1$, given that by the inductive hypothesis we know that $v(\tau(B)) = 1$.

$$\begin{aligned} v(\tau(\neg B)) &= v(\tau(B)) \to \neg v(\tau(B)) \\ &= 1 \to 3 \\ &= 3 \end{aligned}$$

For the case where $A = B \to C$, suppose that $v'(A) = T$. Then we know that either (a) $v'(B) = F$ or (b) $v'(C) = T$. For (a) we know that $v(\tau(B)) \neq 1$ and so:

$$\begin{aligned} v(\tau(B \to C) &= v(\tau(B)) \to (v(\tau(B)) \to v(\tau(C))) \\ &= 2/3 \to (2/3 \to v(\tau(C))) \\ &= 1. \end{aligned}$$

For (b) we know that $v(\tau(C)) = 1$ and so:

$$\begin{aligned} v(\tau(B \to C)) &= v(\tau(B)) \to (v(\tau(B)) \to v(\tau(C))) \\ &= v(\tau(B)) \to (v(\tau(B)) \to 1) \\ &= v(\tau(B)) \to 1 \\ &= 1. \end{aligned}$$

Suppose then that $v'(A) = F$. Then we know that $v'(B) = T$ and $v'(C) = F$ – and thus by the induction hypothesis that $v(\tau(B)) = 1$ and $v(\tau(C)) \neq 1$.

$$\begin{aligned} v(\tau(B \to C)) &= v(\tau(B)) \to (v(\tau(B)) \to v(\tau(C))) \\ &= 1 \to (1 \to 2/3) \\ &= 1 \to (2/3) \\ &= 2/3 \end{aligned}$$

And the result follows. □

Theorem 2.0.5. *For all formulas A we have that:*

$$A_1, \ldots, A_n \vdash_{\mathrm{CL}} B \textit{ if and only if } \tau(A_1), \ldots, \tau(A_n) \vdash_{Ł_3} \tau(B).$$

Proof. For the 'only if' direction suppose that $\tau(A_1), \ldots, \tau(A_n) \nvdash_{Ł_3} \tau(B)$. Then we know there is a ternary valuation v such that $v(\tau(A_i)) = 1$ and $v(\tau(B)) \neq 1$. Letting v_3 be the ternary assignment for which v is the unique ternary valuation, and constructing the corresponding boolean assignment v_3', and boolean assignment v' we have by Lemma 2.0.4 that $v'(A_i) = T$ as $v(\tau(A_i)) = 1$, and $v'(B) = F$ as $v(\tau(B)) \neq 1$, and so $A_1, \ldots, A_n \nvdash_{\mathrm{CL}} B$. A similar argument gives us the 'if' direction. □

2.0.2 T_2 – Embedding Orthologic into KTB

Orthologic is a generalization of "quantum logic" investigated in Goldblatt [1993]. The easiest way to see the idea behind orthologic is to look at it semantically – as this allows us to overlook the syntactic oddities of the logic.[5] Say that a structure $\langle X, \perp\rangle$ is an *orthoframe* if X is a non-empty set, and $\perp$ is an irreflexive and symmetric binary relation – an *orthogonality* relation. Whenever $x \perp y$ for all points $y \in Y \subseteq X$ we will write $x \perp Y$. A set $Y \subseteq X$ is $\perp$-*closed* whenever $\forall x \in X$, $x \notin Y$ only if $\exists y \in X$ such that $y \perp Y$ and not $x \perp y$. A structure $\mathcal{M} = \langle X, \perp, V\rangle$ is an *orthomodel* on the frame $\langle X, \perp\rangle$ iff V is a function that assigns to each propositional variable p_i a $\perp$-closed subset of X, $V(p_i)$. Formulas for orthologic are constructed in the standard way out of a set of countably many propositional variables using the connectives '$\wedge$' (conjunction) and '$\sim$' (orthonegation). What it is for a formula A to be true at a point $x \in X$ in a model $\mathcal{M} = \langle X, \perp, V\rangle$ is defined by induction upon the complexity of A as follows.

$$
\begin{array}{rll}
\mathcal{M} \models_x p_i & \text{if and only if} & x \in V(p_i) \\
\mathcal{M} \models_x A \wedge B & \text{if and only if} & \mathcal{M} \models_x A \text{ and } \mathcal{M} \models_x B \\
\mathcal{M} \models_x \sim A & \text{if and only if} & \forall y \in X, \mathcal{M} \models_y A \text{ only if } x \perp y.
\end{array}
$$

Orthomodels can be understood as giving us a description of the results of performing certain tasks during an experiment. Elements of X represent possible outcomes of a number of operations carried out in the performance of some experiment. Elements x and y are orthogonal ($x \perp y$) iff they are *distinct* outcomes of the *same* operation. In this setting propositions are taken to describe physical events, and are identified with the set of outcomes of operations during our experiment in which they are true – i.e. with subsets of X.

The smallest orthologic O is the set of all formulas A which are valid on (the class of) all orthoframes. What we will do now is show that the following translation τ faithfully embeds O into the normal modal logic

[5] From a syntactic point of view, orthologics are actually sets of ordered pairs of formulas $\langle A, B\rangle$ – as opposed to sets of formulas. For more details on this see Goldblatt [1993, p.83]. We will avoid this issue by considering what are referred to as "orthotheorems" in Rawling & Selesnick [2000] – these being the formulas A which are true throughout all orthomodels.

KTB.

$$\tau(p_i) = \Box\Diamond p_i; \quad \tau(A \wedge B) = \tau(A) \wedge \tau(B); \quad \tau(\sim A) = \Box\neg\tau(A).$$

The translation τ given here is an example of a schematic translation which is not variable fixed, in this case the formula $C(p)$ which is used to interpret the propositional variables being the formula $\Box\Diamond p$.

Proposition 2.0.6. *Suppose that $\langle X, \perp\rangle$ is an orthoframe, and $\langle X, R\rangle$ is a Kripke frame such that $x \perp y$ iff not Rxy, and that $V_\perp$ and V_R are valuation functions such that for all points $x \in X$:*

$$\langle X, R, V_R\rangle \models_x \Box\Diamond p_i \iff \langle X, \perp, V_\perp\rangle \models_x p_i.$$

Then for all formulas in the language of orthologic and all points $x \in X$ we have the following.

$$\langle X, R, V_R\rangle \models_x \tau(A) \iff \langle X, \perp, V_\perp\rangle \models_x A.$$

Proof. By induction upon the complexity of A, the only cases of interest being that in the inductive step where $A = \sim B$ for some formula B.

Suppose that $\langle X, \perp, V_\perp\rangle \models_x \sim B$ for some formula B. This is the case iff for all y, if $\langle X, \perp, V_\perp\rangle \models_y B$ then $x \perp y$. By the inductive hypothesis this is the case iff for all y, if $\langle X, R, V_R\rangle \models_y \tau(B)$ then not Rxy. Taking the contrapositive this is the case iff for all $y \in X$, if Rxy then $\langle X, R, V_R\rangle \not\models_x \tau(B)$, and thus $\langle X, R, V_R\rangle \models_x \Box\neg\tau(B)$. □

Theorem 2.0.7 (Goldblatt [1993, p.91]). *For all formulas A we have the following:*

$$A \in \mathrm{O} \text{ if and only if } \tau(A) \in \mathbf{KTB}.$$

Proof. For the 'only if' direction suppose that $\tau(A) \notin$ **KTB**. Then there is a reflexive, symmetric Kripke model $\mathcal{M}_{KTB} = \langle W, R, V\rangle$ such that $\mathcal{M} \not\models_x \tau(A)$. Construct a new orthomodel $\mathcal{M}_O = \langle W, \perp, V_\perp\rangle$ where $\perp = W \times W \setminus R$ and $V_\perp(p_i) = \{x | \mathcal{M}_{KTB} \models_x \Box\Diamond p_i\}$. Then by Proposition 2.0.6 we know that $\mathcal{M}_O \not\models_x A$ and hence that $A \notin \mathrm{O}$.

For the ‘if’ direction suppose that $A \notin \mathrm{O}$. Then there is an orthomodel $\mathcal{M} = \langle X, \perp, V \rangle$ such that $\mathcal{M} \not\models_x A$. Construct a new Kripke model $\mathcal{M}_{KTB} = \langle X, R, V \rangle$ where $R = X \times X \setminus \perp$. What remains to show is that $\mathcal{M} \models_x p_i \iff \mathcal{M}_{KTB} \models_x \Box\Diamond p_i$. The ‘only if’ direction follows from the **B**-axiom. For the ‘if’ direction suppose that $\mathcal{M}_{KTB} \models_x \Box\Diamond p_i$. Then for all $y \in R(x)$ there is a z such $z \in R(y)$ and $\mathcal{M}_{KTB} \models_z p_i$ [and thus $\mathcal{M} \models_z p_i$]. Suppose then that $x \notin V(p_i)$. Then as $V(p_i)$ is an $\perp$-closed subset of X it follows then that there is a point t such that Rxt and for all $w \in X$ $w \in V(p_i)$ iff not Rtw. But as $t \in R(x)$ it follows that there is a point $z \in R(t)$ such that $z \in V(p_i)$. Thus, as $V(p_i)$ is $\perp$-closed it must be that $x \in V(p_i)$ as desired. Thus by Proposition 2.0.6 it follows that $\mathcal{M}_{KTB} \not\models_x \tau(A)$ and consequently that $\tau(A) \notin \mathbf{KTB}$. $\Box$

2.0.3 T_4 – Embedding S4Grz into S4

There are, though, examples of translations which are variable-fixed but not compositional which are used to embed one logic faithfully into another. Consider the following pair of translations:

$\tau_+(p_i)$	$=$	p_i	$\tau_-(p_i)$	$=$	p_i
$\tau_+(\neg A)$	$=$	$\neg\tau_-(A)$	$\tau_-(\neg A)$	$=$	$\neg\tau_+(A)$
$\tau_+(A \wedge B)$	$=$	$\tau_+(A) \wedge \tau_+(B)$	$\tau_-(A \wedge B)$	$=$	$\tau_-(A) \wedge \tau_-(B)$
$\tau_+(A \to B)$	$=$	$\tau_-(A) \to \tau_+(B)$	$\tau_-(A \to B)$	$=$	$\tau_+(A) \to \tau_-(B)$
$\tau_+(\Box A)$	$=$	$\Box(\Box(\tau_+(A) \to \Box\tau_-(A)) \to \tau_+(A))$	$\tau_-(\Box A)$	$=$	$\Box\tau_-(A)$

We can view the above pair of translations as a single translation by treating τ_- as an auxiliary translation used in the definition of τ_+. Indeed this is the way in which this pair of translations is used to show that we can faithfully embed **S4Grz** into **S4**. Moreover, we can show that the translation τ_+ is variable-fixed but not compositional. To see this consider that if τ_+ were compositional then $\tau_+(\neg\Box p)$ would be $C(\tau_+(\Box p))$ for some formula $C(p)$. But $\tau_+(\neg\Box p) = \neg\Box p$, while $\tau_+(\Box p) = \Box(\Box(p \to \Box p) \to p)$. Therefore, as $\tau_+(\neg\Box p) \neq C(\Box(\Box(p \to \Box p) \to p))$ it follows that τ_+ is not compositional. Similar reasoning can be used to show that τ_- is also not compositional.

To see that this is a recursive translation, the interesting part is the clause for $\Box$ which is that $\tau_+(\Box A) = \Box(\Box(\tau_+(A) \to \Box\tau_-(A)) \to \tau_+(A))$. In this case we have $\Box^\tau(p,q) = \Box(\Box(p \to \Box q) \to p)$ with $m_1 = 2$, $\tau'_{1,1} = \tau_+$ and $\tau'_{1,2} = \tau_-$.

Demri and Goré show that this translation faithfully embeds **S4Grz** into **S4**, making use of the following Lemma.

Lemma 2.0.8. *For all formulas A we have that $A \leftrightarrow \tau_+(A) \in$ **S4Grz** and $A \leftrightarrow \tau_-(A) \in$ **S4Grz**.*

Proof. By simultaneous induction upon the complexity of A – the main cases of interest being those where (i) $A = \Box B$ or (ii)$A = \neg B$.

For case (i) we need to show that (a) $\Box B \leftrightarrow \tau_+(\Box B) \in$ **S4Grz**, and (b) that $\Box B \leftrightarrow \tau_-(\Box B) \in$ **S4Grz** given that both $B \leftrightarrow \tau_-(B) \in$ **S4Grz** and also that $B \leftrightarrow \tau_+(B) \in$ **S4Grz**. For case (a) we know that the following formula is provable in **S4Grz**:

$$[\Box(\Box(p \to \Box p)) \to p] \leftrightarrow \Box p. \tag{2.3}$$

So it follows by uniform substitution that $[\Box(\Box(B \to \Box B)) \to B] \leftrightarrow \Box B \in$ **S4Grz**, and thus by the induction hypothesis $[\Box(\Box(\tau_+(B) \to \Box\tau_-(B))) \to \tau_+(B)] \leftrightarrow \Box B \in$ **S4Grz**. For case (b) we know that $\Box B \leftrightarrow \Box B \in$ **S4Grz**, and thus by the induction hypothesis $\Box B \leftrightarrow \Box\tau_-(B) \in$ **S4Grz** as desired.

For case (ii) we know that $\neg B \leftrightarrow \neg B \in$ **S4Grz**, and thus by the induction hypothesis it follows that both $\neg B \leftrightarrow \neg\tau_-(B) \in$ **S4Grz**, and also that $\neg B \leftrightarrow \neg\tau_+(B) \in$ **S4Grz** as desired. $\Box$

Theorem 2.0.9 (Demri & Goré [2000]). *For all formulas A we have the following:*

$$A \in \mathbf{S4Grz} \textit{ if and only if } \tau_+(A) \in \mathbf{S4}.$$

Proof. For the 'if' direction suppose that $\tau_+(A) \in$ **S4**. Then as **S4** $\subseteq$ **S4Grz** it follows that $\tau_+(A) \in$ **S4Grz** and hence that $A \in$ **S4Grz** by Lemma 2.0.8.

The 'only if' direction is proved using sequent-calculi for **S4** and **S4Grz** in Demri & Goré [2000, p.157f.]. $\Box$

Another interesting example of variable-fixed but not compositional translations is that given in Fitting [1988]. There it is shown that we can faithfully embed **K4** into **K** using the family of translations τ_+^n for $n \in Nat$, which we give below.

$$\begin{array}{lcl|lcl}
\tau_+^n(p_i) & = & p_i & \tau_-^n(p_i) & = & p_i \\
\tau_+^n(\neg A) & = & \neg\tau_-^n(A) & \tau_-^n(\neg A) & = & \neg\tau_+^n(A) \\
\tau_+^n(A \wedge B) & = & \tau_+^n(A) \wedge \tau_+^n(B) & \tau_-^n(A \wedge B) & = & \tau_-^n(A) \wedge \tau_-^n(B) \\
\tau_+^n(A \rightarrow B) & = & \tau_-^n(A) \rightarrow \tau_+^n(B) & \tau_-^n(A \rightarrow B) & = & \tau_+^n(A) \rightarrow \tau_-^n(B) \\
\tau_+^n(\Box A) & = & \Box\tau_+^n(A) & \tau_-^n(\Box A) & = & \Box\tau_-^n(A) \wedge \ldots \wedge \Box^n\tau_-^n(A).
\end{array}$$

What Fitting shows is that $A \in$ **K4** iff there is an $n \in Nat$ such that $\tau_+^n(A) \in$ **K**. The work in Fitting [1988] is elaborated on in Cerrito & Mayer [1997], where some upper bounds on n are given for this translation – in particular they show that $A \in$ **K4** iff $\tau_+^{md(A)^2} \in$ **K** where $md(A)$ is the modal degree of A. Translational embeddings of this kind will be outside the scope of this thesis. Throughout we are concerned with results where we can produce a translation τ and two logics S and S' and use the translation to faithfully embed one into the other. The above kind of result is one where we are given a family $\{\tau_n | n \in Nat\}$ of translations and two logics S and S' such that for any given formula A, there is a translation τ_n in the set such that $\tau_n(A) \in \mathsf{S}'$ – the translation used being determined by the structure of the formula.

2.0.4 T_5 – Embedding Data Logic into Modal Logic

In Veltman [1981] we are presented with a logic designed to model reasoning in environments when the amount of information we can have at any one time is incomplete. The logic in question is designed to model the inferential behaviour of the natural language condition, in addition to the natural language modalities MAY and $MUST$. The semantic structures for this logic are Data Models. A *Data Model* $M = \langle I, \sqsubseteq, V \rangle$ consist of a partially ordered set of points $\langle I, \sqsubseteq \rangle$ each element of which we will think of as an information state, each of whose maximal chains ends in a greatest element – the idea here being that the search for information will eventually result in a complete information state. The function V assigns a partial valuation of the propositional variables V_i to each information state $i \in I$. Additionally the valuation is required to be *persistent* in the sense that whenever $i \sqsubseteq j$ we have that V_j is an extension of V_i.

The reason for having our valuation functions be partial functions (rather than functions) is that it allows us to model the effects of having partial information – of agents being able to have information to the effect that 'A' is true without having information that '$\sim A$' is false. To this end we need to distinguish between a model failing to tell us that something is true, and a model telling us that something is false. To this end we will write $M \models_x A$ whenever 'A' is true at x (i.e. $V_x(A) = 1$), and $M \dashv_x A$ whenever 'A' is false at x (i.e. $V_x(A) = 0$), noting that we can have both $M \not\models_x A$ and $M \not\dashv_x A$ whenever $V_x(A)$ is undefined.

The propositional language for Data logic consists of the connectives $\sim, \wedge, \vee, \rightarrow, MAY, MUST$, whose semantic clauses are as follows.

$M \models_x p_i$	if	$V_x(p_i) = 1$
$M \dashv_x p_i$	if	$V_x(p_i) = 0.$
$M \models_x A \wedge B$	if	$M \models_x A$ and $M \models_x B$
$M \dashv_x A \wedge B$	if	$M \dashv_x A$ or $M \dashv_x B$
$M \models_x A \vee B$	if	$M \models_x A$ or $M \models_x B$
$M \dashv_x A \vee B$	if	$M \dashv_x A$ and $M \dashv_x B$
$M \models_x \sim A$	if	$M \dashv_x A$
$M \dashv_x \sim A$	if	$M \models_x A$
$M \models_x A \to B$	if	$\forall y(y \sqsupseteq x,\ M \models_y A$ only if $M \models_y B.)$
$M \dashv_x A \to B$	if	$\exists y(y \sqsupseteq x,\ M \models_y A$ and $M \dashv_y B.)$
$M \models_x MAY(A)$	if	$\exists y(y \sqsupseteq x,\ M \models_y A.)$
$M \dashv_x MAY(A)$	if	$\forall y(y \sqsupseteq x,$ not $M \models_y A.)$
$M \models_x MUST(A)$	if	$\forall y(y \sqsupseteq x \& \forall z(z \sqsupseteq y \Rightarrow z = y)$ only if $M \models_y A).$
$M \dashv_x MUST(A)$	if	$\exists y(y \sqsupseteq x \& \forall z(z \sqsupseteq y \Rightarrow z = y)$ and $M \dashv_y A).$

In van Benthem [1986] a translation which faithfully embeds Data Logic into $\mathbf{S4G}_c$ is given, where $\mathbf{S4G}_c$ (sometimes referred to as $S4.1$) is the normal extension of **S4** by the following axiom schema:

$$\mathbf{G}_c: \quad \Box\Diamond A \to \Diamond\Box A.$$

This normal modal logic is determined by the class of frames $\langle W, R\rangle$ such that R is reflexive, transitive and in addition satisfy the condition that $\forall x \exists y(Rxy \wedge \forall z(Ryz \to y = z))$.

The translation in question maps formulas in the data logic language to formulas in the modal language, and is defined as follows.

$(p_i)^+$	$=$	$\Box p_i$	$(p_i)^-$	$=$	$\Box\neg p_i$
$(\sim A)^+$	$=$	$(A)^-$	$(\sim A)^-$	$=$	$(A)^+$
$(A \wedge B)^+$	$=$	$(A)^+ \wedge (B)^+$	$(A \wedge B)^-$	$=$	$(A)^- \vee (B)^-$
$(A \vee B)^+$	$=$	$(A)^+ \vee (B)^+$	$(A \vee B)^-$	$=$	$(A)^- \wedge (B)^-$
$(A \to B)^+$	$=$	$\Box((A)^+ \to (B)^+)$	$(A \to B)^-$	$=$	$\Diamond((A)^+ \wedge (B)^-)$
$(MAY(A))^+$	$=$	$\Diamond(A)^+$	$(MAY(A))^-$	$=$	$\Box\neg(A)^+$
$(MUST(A))^+$	$=$	$\Box\Diamond(A)^+$	$(MUST(A))^-$	$=$	$\Diamond\Box(A)^-$

At first glance it may appear as if $(\cdot)^+$ is a compositional translation, but as it happens when we remove the double recursion in the clauses for $\sim$ we can see (as noted in Escriba [1989]) that this is in fact a non-compositional translation. For this translation to be compositional then we would have $(\sim p)^+ = \#(p)^+$. We know that $p^+ = \Box p$, and that $(\sim p)^+ = (p)^- = \Box\neg p$ – but that for no formula # do we have $\#(\Box p) = \Box\neg p$. So $(\cdot)^+$ is not compositional.

Now, in order to show that this translation is a faithful embedding of Data Logic into $\mathbf{S4G_c}$ we will need to describe two different model constructions: one which takes us from a data model to a model for $\mathbf{S4G_c}$ and one which takes us from a model for $\mathbf{S4G_c}$ to a Data Model.

Proposition 2.0.10 (van Benthem [1986, p.234]). *Let $M = \langle I, \sqsubseteq, V\rangle$ be a Data Model, and construct a new model $\mathcal{M} = \langle I, \sqsubseteq, V^*\rangle$ by setting $V^*(p_i) = \{i \in I | V_i(p_i) = 1,$ or $V_i(p_i) = 0$ and for some $j \sqsupseteq i$ $V_j(p_i) = 0\}$. Then for all point $i \in I$ and data logic formulas A we have the following.*

$$M \models_i A \quad \text{if and only if} \quad \mathcal{M} \models_i (A)^+ \tag{2.4}$$

$$M \dashv_i A \quad \text{if and only if} \quad \mathcal{M} \models_i (A)^-. \tag{2.5}$$

Proof. By induction upon the complexity of A, the only case of interest being the basis case.

For the 'only if' direction of (2.4) suppose that $M \models_i p_i$. Then we know that $V_i(p_i) = 1$, and hence as V is hereditary that for all $j \sqsubseteq i$ that $V_j(p_i) = 1$. Hence we know that for all $j \sqsubseteq i$ that $V^*(p_i) = 1$ and thus that $\mathcal{M} \models_i \Box p_i$. Suppose then for the 'if' direction that $\mathcal{M} \models_i \Box p_i$. Then it follows that

$\mathcal{M} \models_i p_i$ and thus by the definition of V^* either $V_i(p_i) = 1$ (and hence $M \models_i p_i$) or $V_i(p_i) \neq 0$ and there is a $j \sqsubseteq i$ such that $V_j = 0$ – but $\mathcal{M} \models_j p_i$, and hence $V_j(p_i)$ cannot be 0!

For the 'only if' direction of (2.5) suppose now that $M \dashv_i p_i$. Then $V_i(p_i) = 0$ and $V_j(p_i) = 0$ for all $j \sqsubseteq i$ – so for all such j we have that $\mathcal{M} \models_j \neg p_i$ – and thus $\mathcal{M} \models_i \Box\neg p_i$. Suppose then for the 'if' direction that $\mathcal{M} \models_i \Box\neg p_i$. Consider now any maximal $j \sqsubseteq i$. It follows then that $\mathcal{M} \models_j \neg p_i$, and hence $V_j(p_i) = 0$. Suppose then that $V_i(p_i) \neq 0$. Then we would have that $V_i^*(p_i) = 1$ and thus $\mathcal{M} \models_i p_i$ and also $\mathcal{M} \models_i \neg p_i$! So $V_i(p_i) = 0$ and thus $M \dashv_i p_i$. □

Proposition 2.0.11 (van Benthem [1986, p.235]). *Let $\mathcal{M}' = \langle W, R, V \rangle$ be a model for* $\mathbf{S4G_c}$*, and construct a new model $M' = \langle W, R, V' \rangle$ where $V'_x(p_i) = 1$ if $\mathcal{M}' \models_x \Box p$ and $V'_x(p_i) = 0$ if $\mathcal{M}' \models_x \Box\neg p$. Then for all points $i \in W$ and data logic formulas A we have the following.*

$$M' \models_i A \quad \textit{if and only if} \quad \mathcal{M}' \models_i (A)^+ \tag{2.6}$$

$$M' \dashv_i A \quad \textit{if and only if} \quad \mathcal{M}' \models_i (A)^-. \tag{2.7}$$

Theorem 2.0.12 (van Benthem [1986, p.235]). *A is a theorem of Data Logic if and only if A^+ is a theorem of* $\mathbf{S4G_c}$.

2.0.5 Non-Recursive Translations

As it turns out we will need one more property in order to properly characterize the class of non-recursive translations, and give our final taxonomy of translations. The easiest way to introduce this property is by way of two examples – in this case looking at the T_7-translations and T_9-translations.

2.0.5.1 T_7 – Translations and the Admissibility of Rules

Translations which are variable-fixed but not compositional can be of great use in giving syntactic proofs of the admissibility of rules in modal logics.

We can find examples of this enterprise in Williamson [1993], which is inspired by similar syntactic results in Chellas [1980, p.124f.]. The syntactic method of proving rule admissibility using translations allows us to prove a number of results where the usual model theoretic methods are not applicable. The translations used in the above two references are examples of what M. Crabbé calls 'normal transformations' in Crabbé [1991]. A *normal transformation* is a translation Φ which maps formulas to formulas such that:

- $\Phi(A) = A$ if A is p_i or $\top$ or $\bot$.
- $\Phi(\neg A) = \neg\Phi(A)$.
- $\Phi(A \to B) = \Phi(A) \to \Phi(B)$.

In particular we will be concerned with normal tranformations constructed out of the normal transformation $\Phi_C(\Box A) = C$ for some formula C.

Consider now the following normal translations which we will use below to show that Löb's rule is admissible in **K**. Given a normal transformation Φ define the following sequence Φ_n of normal transformations:

- $\Phi_0 = \Phi$
- $\Phi_{n+1}(\Box A) = \Box\Phi_n(A)$.

Lemma 2.0.13. *For all $n \geq comp(A)$ we have that $\Phi_n(A) = A$.*

In particular we will be concerned with the translation $\Phi_{\top,n}$ where:

- $\Phi_{\top,n}(A) = A$ for all $n \geq comp(A)$.
- $\Phi_{\top,n+1}(\Box A) = \Box^n\top$.

Lemma 2.0.14. $\Phi_{\top,n}(\Box^{n+1}A) = \Box^n\top$.

Proposition 2.0.15. *The translation $\Phi_{\top,n}$ embeds* **K** *into itself.*

Given all of this we are now in a position to give a syntactic proof of the admissibility of Löb's rule in **K**.

Theorem 2.0.16 (Crabbé). *If* $\Box A \to A \in \mathbf{K}$ *then* $A \in \mathbf{K}$.

Proof. Suppose that $\Box A \to A \in \mathbf{K}$ and fix on an n such that $n \geq comp(A)$. Then as $\Box A \to A \in \mathbf{K}$ it follows by monotonicity and modus ponens that $\Box^{n+1} A \to A \in \mathbf{K}$. By Proposition 2.0.15 we know that $\Phi_{\top,n}(\Box^{n+1} A \to A) \in \mathbf{K}$, i.e. that $\Box^n \top \to A \in \mathbf{K}$. As $\Box^n \top \in \mathbf{K}$ it follows by modus ponens that $A \in \mathbf{K}$ as desired. $\Box$

2.0.5.2 T_9 – One-off Translations

After definitional translations the next most common translation are those which apply a 'one-off' non-recursive manipulation to formulas. The classic example here is Glivenko's translation which faithfully embeds the set of classical theorems into the set of intuitionistic theorems in the following sense.

Theorem 2.0.17 (Glivenko). *For all formulas A we have the following:*

$$A \in CL \text{ if and only if } \neg\neg A \in IL.$$

One-off translations are those translations τ where $\tau(A)$ is the formula $C(A)$ for some formula $C(p)$ constructed out of at least the propositional variable p. Note that this is a more liberal use of the $C(\cdot)$ notation than used earlier, where the formula C was forced to be constructed solely out of the propositional variable p. In the Glivenko case the formula C here is of this more restricted type, being the formula '$\neg\neg p$', but the (admittedly not very useful) translation for which $\tau(A) = A \wedge q$ where the formula $C(p) = p \wedge q$ would also be counted as a 'one off' translation.

Consider the translation τ' which is homonymous on the classical connectives for which we also have:

$$\tau'(p_i) = p_{i+1}; \quad \tau'(\Box A) = \Box\tau'(A) \wedge p_0. \tag{2.8}$$

The translation mentioned above is a T_5-translation. But now consider the translation τ such that $\tau(A) = p_0 \to \tau'(A)$. This translation is used in Aanderaa [1969] to faithfully embed **S2** into **KT**. We mention this translation here mostly to point out that it is not, despite what one might briefly think, a 'one-off' translation. As it happens this is in fact a T_5-translation – being recursive, non-compositional and not variable-fixed.

What these two example allows us to highlight is a particular property of translations – namely that of a translation being *depth-limited*. We can broadly think of non-recursive translations as coming in two varieties – those which are *one-off* translations and those which are *depth-limited*. A translation τ is a *one-off* translation if the result of translating a formula A is just the result of substituting that formula into another fixed formula – i.e. one where $\tau(A) = C(A)$ for some formula $C(p)$. The *depth-limited* translations are those for which we can isolate the failure of recursiveness in a particular translation clause. In the case above this is the clause for $\Box A$ which is not of the form $C(\tau(A))$ for some formula C. Given this, we are left with the Taxonomy depicted in Figure 2.3

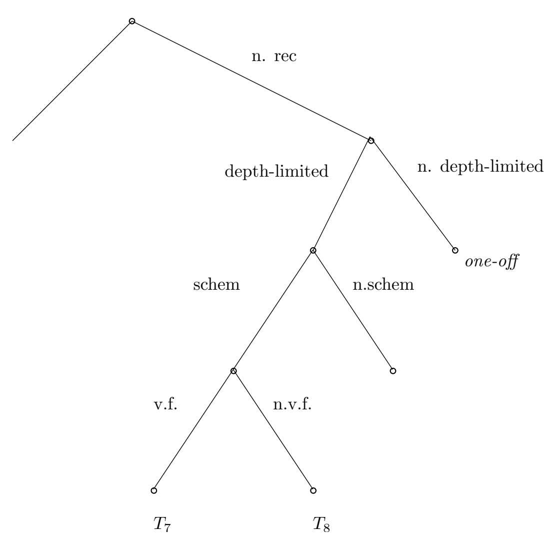

Figure 2.3: A Reclassification of the Non-Recursive Translations.

There are just a few housecleaning notes left to be made regarding our final taxonomy. Firstly we coudl

Firstly, we could derive all of the above results concerning the admissibility of Löb's rule in **K** in terms of translations for which $\Phi(p_i) = \neg\neg p_i$ this being an example of a T_6 translation in our taxonomy. We can also get an example of a T_6 translation by altering our example of a T_4 translation above so that $\tau_+(p_i) = p_{i+1}$ and $\tau_-(p_i) = p_{i+1}$.

Secondly, we should technically split our 'one-off' translations into two different classes – those which are schematic and those which are not. We have not done so in the interests of brevity. What the schematic/non-schematic distinction tells us in this case is what the structure of the formula used by the translation is. If the formula is constructed out of a single variable p then the translation will be schematic. It is worth mentioning that the translation which we introduced at the start of this chapter falls into the non-schematic one-off translation camp.

Lastly we turn to an interesting example of a translation which is both non-recursive and non-schematic – the unnamed category in our new taxonomy.

2.0.6 Non-Recursive, Non-Schematic Translations: Embedding Johansson's Minimal Logic into Intuitionistic Logic

The following translation, a variation of which appears in Prawitz & Malmnäs [1968, p.219], embeds Johansson's Minimal Logic (J) into Intuitionistic Logic (IL). The idea here is to single out a propositional variable (p_0) to act as the falsum constant $\perp$, which minimal logic treats as an arbitrary propositional constant. To accommodate this we 'shift' the propositional variables used in a formula so that wherever the original formula talks about p_i the translation talks about p_{i+1}.

$$\tau(p_i) = p_{i+1}; \quad \tau(\bot) = p_0; \quad \tau(A \to B) = \tau(A) \to \tau(B);$$

It is easy to see that this translation is not schematic (and hence also not variable fixed). To see that it is not recursive what we need to note is that, in order for the translation to count as a recursive one we would need to have a formula $\bot^\tau$ constructed out of the empty set of propositional variables, which p_0 is not.

Translations which use this method of getting a propositional variable to act as a propositional constant appear in the literature on modal logic also – being mentioned in a footnote as a modification of a translation given in Cresswell [1967, p.201] to faithfully embed Lemmon's non-normal modal logic **E2** (i.e. **EMT**) into **KT** – the propositional variable being used to stand in for a propositional constant true at exactly the normal worlds in a model – the translation clause for $\Box A$ then becoming '$p_0 \wedge \Box\tau(A)$'.

We will prove the following embedding result semantically by making use of the semantics for minimal and intuitionistic logic given in Segerberg [1968]. Let us say that a set X is *hereditary* under a relation R whenever if $x \in X$ and Rxy then $y \in X$. Models here are structures $\langle X, R, Q, V\rangle$ where X is a nonempty set, R is a partial ordering on X, $Q \subseteq X$ is a set of non-normal elements which is hereditary under R, and V is a function which maps the propositional variables to hereditary subsets of X. Truth of a formula at a point x in a model $\mathcal{M} = \langle X, R, Q, V\rangle$ is defined inductively as follows.

$$\begin{array}{rll} \mathcal{M} \models_x p_i & \text{if and only if} & x \in V(p_i) \\ \mathcal{M} \models_x \bot & \text{if and only if} & x \in Q \\ \mathcal{M} \models_x A \to B & \text{if and only if} & \forall y([Rxy \text{ and } \mathcal{M} \models_y A] \Rightarrow \mathcal{M} \models_y B) \end{array}$$

Say that a model $\langle X, R, Q, V\rangle$ is *normal* whenever $Q = \emptyset$. Then (for the present purpose) IL will denote the logic determined by the class of all normal models, and J the logic determined by the class of all models.[6]

Then IL is determined by the class of all normal models, and J by the class of all models.

Theorem 2.0.18. *For all formulas A we have that:*

$$\vdash_{\mathrm{J}} A \textit{ if and only if } \vdash_{\mathrm{IL}} \tau(A).$$

Proof. For the 'if' direction suppose that $\nvdash_{\mathrm{IL}} \tau(A)$. Then there is a normal model $\mathcal{M} = \langle X, R, V\rangle$ such that $\mathcal{M} \nvDash_x \tau(A)$ for some point $x \in X$. Construct a new model $\mathcal{M}' = \langle X, R, Q, V'\rangle$ by setting $Q = \{x | x \in V(p_0)\}$ and $V'(p_i) = V(p_{i+1})$. Then it is quick to prove by induction upon the complexity of formulas that $\mathcal{M} \vDash_x \tau(A) \iff \mathcal{M}' \vDash_x A$, and thus that $\mathcal{M}' \nvDash_x A$. As this is a model for J it follows then that $\nvdash_{\mathrm{J}} A$.

For the 'only if' direction suppose that $\nvdash_{\mathrm{J}} A$. Then there is a model $\mathcal{M} = \langle X, R, Q, V\rangle$ such that $\mathcal{M} \nvDash_x A$ for some point $x \in W$. Construct a new model $\mathcal{M}' = \langle X, R, \emptyset, V'\rangle$ by setting $V'(p_i) = V(p_{i-1})$ for all $i > 0$ and $V'(p_0) = Q$. It is again easy to show by induction upon the complexity of formulas that $\mathcal{M} \vDash_x A \iff \mathcal{M}' \vDash_x \tau(A)$, and hence that $\mathcal{M}' \nvDash_x \tau(A)$ – from which it follows that $\nvdash_{\mathrm{IL}} \tau(A)$. $\square$

2.0.7 Translations and Substitutions

In this section we will briefly look at the interactions between translations and substitutions – in particular investigating a commutativity condition which relates translations and substitutions which we will use implicitly from here on. Before continuing it will be useful to recall the definition of what it is for a function σ be be a substitution.

Definition 2.0.19. A function $\sigma : L \to L$ (where L is a propositional language) is a *substitution* if and only if $\sigma\#(A_1, \ldots, A_n) = \#(\sigma(A_1), \ldots, \sigma(A_n))$ for every n-ary connective # in the language under consideration.

[6]Technically we are only dealing with the embedding of the $\{\to, \bot\}$-fragments of both IL and J, although nothing essential hangs on this.

The following observation is due to Lloyd Humberstone (unpublished).

Theorem 2.0.20. *Suppose that τ is a compositional and variable-fixed translation from a logic $\vdash_0$ to a logic $\vdash_1$. Then for all substitutions $\sigma : L_0 \to L_0$, there exists a substitution $\sigma' : L_1 \to L_1$ such that:*

$$\sigma' \circ \tau = \tau \circ \sigma. \tag{2.9}$$

Proof. Let $\sigma'(p_i) = \tau(\sigma(p_i))$. Let $A = \#(p_1, \ldots, p_n)$ where $\#$ is some primitive connective of L_0. Then the rhs of the above inset equation is

$$\#^{\tau}(\tau(\sigma(p_1)), \ldots, \tau(\sigma(p_n))).$$

This is clearly the result of substituting $\tau(\sigma(p_i))$ for p_i in $\#^{\tau}$ – which is to say $\sigma'(\tau(A))$, and the result follows. □

One might wonder whether the converse to the above Theorem holds, namely that if a translation τ is such that, for every substitution σ there is a substitution σ' such that $\sigma' \circ \tau = \tau \circ \sigma$, then τ is variable fixed and compositional. To see that this isn't the case consider the following example.

Proposition 2.0.21. *For all substitutions σ, and and the 'one off' translation $\tau(A) = \neg\neg A$, $\sigma \circ \tau = \tau \circ \sigma$.*

Proof. We will show that $\sigma(\tau(A)) = \tau(\sigma(A))$. To do this we note that the lhs is $\neg\neg\sigma(A)$. The rhs is simply $\sigma(\neg\neg A)$, which by the definition of a substitution is just $\neg\neg\sigma(A)$, completing the proof. □

As we can see, the essential property which allows this to happen is the way in which substitutions interact with the connectives – acting homonymously upon them. In fact, it is quite easy to see that we can prove the analogous result for any 'one-off' translation. The pertinent fact isolated here is that the property of having substitutions interact with translations in the way specified above does not ensure that the translation is compositional – this being the property which our counterexample capitalized upon. This then raises the following question: when exactly is it the case that we have that a translation τ has the property that for all substitutions σ there is a substitution σ' such that $\tau \circ \sigma = \sigma' \circ \tau$.

As the above counterexample shows, clearly the translation being compositional is not what's important. It is equally easy to show that it is not variable-fixedness which is important here. To show this we will return to the translations we have mentioned above for which $\tau(p_i) = p_{i+1}$.

Theorem 2.0.22. *Suppose that τ is a translation such that $\tau(p_i) = p_{i+1}$, and for all primitive connectives # of L_0 we have that $\tau(\#(A_1,\ldots,A_n) = B(\tau(A_1),\ldots,\tau(A_n),p_0)$ for some formula B in L_1. Then for all substitutions σ, there is a translation σ' such that $\sigma' \circ \tau = \tau \circ \sigma$.*

Proof. Define σ' as follows.

$$\sigma'(p_i) = \begin{cases} p_0, & i = 0 \\ \tau(\sigma(p_{i-1})), & i > 0. \end{cases}$$

Let $A = \#(p_0,\ldots,p_n)$. Then the rhs of the condition is

$$B(\tau(\sigma(p_0)),\ldots,\tau(\sigma(p_n)),p_0).$$

For the lhs we have

$$\sigma'(B(p_0,\ldots,p_n,p_{n+1}) = B(p_0,\tau(\sigma(p_0)),\ldots,\tau(\sigma(p_{n-1})),\tau(\sigma(p_n))),$$

and the result follows. $\square$

Translations like those in the above theorem are interesting in that they are, in many respects, very nearly definitional while still failing to be compositional or variable fixed. Two interesting things come out of the above discussion. The first is that we are able to construct substitutions which interact with translations in the desired fashion for a number of different kinds of translations. The second, and perhaps more interesting is that it shows that the taxonomy outlined above does not capture all the various interesting properties of translations in the sense that the features of translations used to construct the taxonomy are not all and only the features which are important in assessing whether a translation has a particular property.

2.1 Modal Translations

Throughout the rest of this thesis we will focus on a particular kind of definitional translation, although we will still have occasion to mention other kinds of translations. There is a good philosophical reason to focus on definitional translations. The presence of a definitional translation between two logics carries with it some immediate information, namely that we can define all of the connectives of the source logic in terms of the target logic – the clauses of the translation giving us a recipe as to how. In what follows we will mainly be considering modal logics, and as all of these leave the classical connectives unchanged we will be concerned with translations which reflect this fact – translating the classical connectives as themselves.

Definition 2.1.1. Let $C(p)$ be a formula of at most one variable p. Then the translation τ_C is defined as follows: $\tau_C(p_i) = p_i$, $\tau_C(\#(A_1, \ldots, A_n)) = \#(\tau_C(A_1), \ldots, \tau_C(A_n))$ for all classical connectives $\#$, and $\tau_C(\Box A) = C(\tau_C(A))$.

Definition 2.1.2. A translation τ is a *modal-to-modal* translation iff τ is τ_C for some formula $C(p)$ of at most one variable.

We will follow Zolin [2000] in thinking of a formula $C(p)$ as inducing the *modality* $\lambda p.C(p)$ – both the formula and modality being referred to as $C(p)$, although often we will also refer to $\lambda p.C(p)$ simply as C. Such formulas are called *singulary modal functions* in Hughes & Cresswell [1968]. What Hughes and Cresswell refer to as a modality (a sequence of $O_1 \ldots O_n$ where each O_i is either $\Box$ or $\neg$) Zolin calls a *linear modality*. Given a modal logic S and a modality $C(p)$, we will (following Zolin [2000]) define the logic of $C(p)$ over S – $\mathsf{S}(C)$ – as the following set of formulas.

$$\mathsf{S}(C) = \{A|\tau_C(A) \in \mathsf{S}\}. \tag{2.10}$$

It is easy to see that the translation τ_C faithfully embeds S into S' whenever $\mathsf{S}'(C) = \mathsf{S}$. We will say that two modalities C' and C'' are *equivalent over a logic* S whenever $C'(p) \leftrightarrow C''(p) \in \mathsf{S}$, and *analogous over* S whenever $\mathsf{S}(C') = \mathsf{S}(C'')$.

To illustrate the notation introduced above we will look at a very simple modal-to-modal translation, the translation $\tau_{\Box\Box}$ which replaces all occurrences of $\Box$ within a formula with $\Box\Box$. Say that a modal logic S is *iterative* if $\mathsf{S}(\Box\Box) = \mathsf{S}$ – i.e. if $\Box$ and $\Box\Box$ are analogous over S. In Zolin [2000] it is shown that **KTB** is iterative. What we will now show is that **K** is iterative, using a different model construction from that given in Humberstone [2006] to a somewhat similar effect, the difference arising concerning how reflexive points are treated.

Definition 2.1.3. Given a model $\mathcal{M} = \langle W, R, V\rangle$, let $\overline{W} = \{\overline{w}|w \in W\}$ be a set of points disjoint from, and in one-to-one correspondence with W, and construct a new model $\mathcal{M}^+ = \langle W^+, S, V^+\rangle$ as follows.

- $W^+ := W \cup \overline{W}$.
- $S := \{\langle x, \overline{x}\rangle|x \in W\} \cup \{\langle \overline{x}, y\rangle|Rxy\} \cup \{\langle x, x\rangle, \langle \overline{x}, \overline{x}\rangle|Rxx\}$.
- $V^+(p_i) := V(p_i) \cup \{\overline{x}|x \in V(p_i)\}$.

What the above model construction does is add new points ('understudies') to our original model, and alter the accessibility relation so that for all points $x \in W$ any point in $R(x)$ is now in $S(\overline{x})$ – with both the understudy and the original point being reflexive in the new model whenever the original point was reflexive in the original model.

Lemma 2.1.4. *Let $\mathcal{M}^+$ be the result of applying the above model construction to a model $\mathcal{M}$. Then for all formulas A, and all points $x \in W$ we have the following.*

$$\mathcal{M}^+ \models_x \tau_{\Box\Box}(A) \text{ if and only if } \mathcal{M}^+ \models_{\overline{x}} \tau_{\Box\Box}(A).$$

Proof. By induction upon the complexity of A. For the base case where A is a propositional variable p_i we note that the construction of V^+ ensures that $\mathcal{M}^+ \models_x p_i \iff \mathcal{M}^+ \models_{\overline{x}} p_i$. The only case of interest then is that in the inductive step where $A = \Box B$ for some formula B.

For the 'only if' direction assume for a contradiction that both $\mathcal{M}^+ \models_x \Box\Box\tau_{\Box\Box}(B)$ and $\mathcal{M}^+ \not\models_{\overline{x}} \Box\Box\tau_{\Box\Box}(B)$. This means that either (i) $S\overline{xx}$ and $\mathcal{M}^+ \not\models_{\overline{x}} \tau_{\Box\Box}(B)$, (ii) $\mathcal{M}^+ \not\models_y \tau_{\Box\Box}(B)$, for some $y \in S(\overline{x})$, or (iii) $\mathcal{M}^+ \not\models_{\overline{z}} \tau_{\Box\Box}(B)$ for some $\overline{z} \in S(S(\overline{x}))$. If (i) is the case then by the induction hypothesis we know that $\mathcal{M}^+ \not\models_x \tau_{\Box\Box}(B)$ and hence as $S\overline{xx}$ that Sxx, and thus that $\mathcal{M}^+ \not\models_x \Box\Box\tau_{\Box\Box}(B)$. For (ii) we need to note that, by construction, anything which is 1 S-step away from $\overline{x}$ is 2 S-steps way from x, and hence that $\mathcal{M}^+ \not\models_y \tau_{\Box\Box}(B)$. Consequently it follows that $\mathcal{M}^+ \not\models_x \Box\Box\tau_{\Box\Box}(B)$. For (iii) we note that by the induction hypothesis this would require that $\mathcal{M}^+ \not\models_z \tau_{\Box\Box}(B)$. As $\overline{z}$ is 2 S-steps away from $\overline{x}$ we know that z is 2 S-steps away from x, and consequently that $\mathcal{M}^+ \not\models_x \Box\Box\tau_{\Box\Box}(B)$. As all three cases lead to a contradiction we can conclude that $\mathcal{M}^+ \models_{\overline{x}} \Box\Box\tau_{\Box\Box}(B)$.

For the 'if' direction now assume that $\mathcal{M}^+ \not\models_x \Box\Box\tau_{\Box\Box}(B)$ and that $\mathcal{M}^+ \models_{\overline{x}} \Box\Box\tau_{\Box\Box}(B)$. This would mean that either (i) Sxx and $\mathcal{M}^+ \not\models_x \tau_{\Box\Box}(B)$, (ii) $S\overline{xx}$ and $\mathcal{M}^+ \not\models_{\overline{x}} \tau_{\Box\Box}(B)$ or (iii) $\mathcal{M}^+ \not\models_y \tau_{\Box\Box}(B)$ for some $y \in S(S(x))$. For (i) and (ii) it is enough to note that for these to be the case we must have that Sxx and $S\overline{xx}$. For (i) we know by the induction hypothesis that $\mathcal{M}^+ \not\models_{\overline{x}} \tau_{\Box\Box}(B)$ (which is (ii)), and hence as $S\overline{xx}$ that $\mathcal{M}^+ \not\models_{\overline{x}} \Box\Box\tau_{\Box\Box}(B)$. For (iii) we note that by the induction hypothesis this would mean that $\mathcal{M}^+ \not\models_{\overline{y}} \tau_{\Box\Box}(B)$, and hence as this point is 2 S-steps away from $\overline{x}$ that $\mathcal{M}^+ \not\models_{\overline{x}} \Box\Box\tau_{\Box\Box}(B)$. As all three cases lead to a contradiction we can conclude that $\mathcal{M}^+ \models_x \Box\Box\tau_{\Box\Box}(B)$. $\Box$

Theorem 2.1.5. *For all formulas A and all points $x \in W$ we have.*

$$\mathcal{M} \models_x A \text{ if and only if } \mathcal{M}^+ \models_x \tau_{\Box\Box}(A).$$

Proof. By induction on the complexity of A, the only case of interest being in the inductive step where $A = \Box B$.

For the 'only if' direction suppose that $\mathcal{M} \models_x \Box B$. So it follows that for all points $y \in R(x)$ that $\mathcal{M} \models_y B$. By the induction hypothesis we have that $\mathcal{M}^+ \models_y \tau_{\Box\Box}(B)$. Suppose, then, that is is not the case that $\overline{x}$ is reflexive. Then we know that $R(x) = S(\overline{x})$, and thus that $\mathcal{M}^+ \models_{\overline{x}} \Box\tau_{\Box\Box}(B)$. In this case as $\overline{x}$ is reflexive exactly when x is we know that $S(x) = \overline{x}$, and thus that $\mathcal{M}^+ \models_x \Box\Box\tau_{\Box\Box}(B)$ as desired. Suppose now that $\overline{x}$ is reflexive. Then we know that Sxx and hence that Rxx, and thus by the induction hypothesis that $\mathcal{M}^+ \models_x \tau_{\Box\Box}(B)$. By Lemma 2.1.4 it follows that $\mathcal{M}^+ \models_{\overline{x}} \tau_{\Box\Box}(B)$, and hence that $\mathcal{M}^+ \models_{\overline{x}} \Box\tau_{\Box\Box}(B)$. As $S(x) = \{x, \overline{x}\}$, and both of these points validate $\tau_{\Box\Box}(B)$ it follows that $\mathcal{M}^+ \models_x \Box\tau_{\Box\Box}(B)$, and thus that $\mathcal{M}^+ \models_x \Box\Box\tau_{\Box\Box}(B)$.

Suppose now for the 'if' direction that $\mathcal{M} \not\models_x \Box B$. Then there is some point $y \in R(x)$ such that $\mathcal{M} \not\models_x B$. By the induction hypothesis it follows that $\mathcal{M}^+ \not\models_y \tau_{\Box\Box}(B)$. As $y \in R(x)$ it follows that $y \in S(\overline{x})$ and hence that $\mathcal{M}^+ \not\models_{\overline{x}} \Box\tau_{\Box\Box}(B)$. As $Rx\overline{x}$ it thus follows that $\mathcal{M}^+ \not\models_x \Box\Box\tau_{\Box\Box}(B)$ as desired. $\Box$

It is easy to show that the modality $\Box\Box$ is normal in **K**. This fact, coupled with the above result allows us to show the following.

Theorem 2.1.6. **K** *is iterative (i.e.* $\mathbf{K}(\Box\Box) = \mathbf{K}$*).*

In fact, the above model construction has one very useful advantage over that given in Humberstone [2006]. The model construction there took a reflexive model and transformed it into a model where S^2 was reflexive. What our construction does though is take a reflexive model and transform it into a reflexive model. Thus, using the above Theorem and the fact that $\tau_{\Box\Box}(\Box A \to A) \in \mathbf{KT}$ allows us to conclude the following.

Corollary 2.1.7. **KT** *is iterative (i.e.* $\mathbf{KT}(\Box\Box) = \mathbf{KT}$*).*

Corollary 2.1.8. **KD** *is iterative (i.e.* $\mathbf{KD}(\Box\Box) = \mathbf{KD}$*).*

2.1.1 Modal-to-Modal Translations and Philosophy: Some Examples

In this section we will provide some examples motivating the relevance of translations to questions in philosophy.

2.1.1.1 Epistemic Logic: Knowledge and Subjective Certainty

In Hintikka [1962] we are presented with arguments to the effect that the correct logic of knowledge is at least as strong as **S4**, and it is within the culture of these arguments in that the following discussion is set. Most work on attempting to determine the correct formal theory of knowledge and belief considers the formal properties of knowledge and belief in isolation – trying to determine the correct logics of these notions independently. One might argue, though, that our understanding of both of these concepts arise mostly out of the way in which they interact. Given this, one way for us to determine the correct formal theory of knowledge and belief is to consider the formal properties and interactions between them. Given this, consider the following interaction principles laid out in Stalnaker [2006].

(PI)	$Bp \rightarrow KBp$	Positive Introspection
(NI)	$\neg Bp \rightarrow K\neg Bp$	Negative Introspection
(KB)	$Kp \rightarrow Bp$	Knowledge implies Belief
(CB)	$Bp \rightarrow \neg B\neg p$	Consistency of Belief
(SB)	$Bp \rightarrow BKp$	Strong Belief.

The concept of belief intended here is that of "subjective certainty" – called 'conviction' in Lenzen [1979]. That is to say, we are interpreting $B_a\varphi$ (agent a believes that φ) as meaning that $Prob_a(\varphi) = 1$ – where $Prob_a$ is a's subjective probability function. Sis the normal modal logic containing

(PI), (NI), (KB), (CB), (SB) as well as **T** and **4** for the K-operator. [7] Then we can note the following property of our combined doxastic/epistemic logic.

Proposition 2.1.9.

$$Bp \leftrightarrow \neg K \neg K p \in \mathsf{S} \tag{2.11}$$

Proof. The '$\rightarrow$' half follows from (SB), (CB) and (KB), and the '$\leftarrow$' half follows from (KB) and (NI). □

This gives us a way of defining the belief operator in terms of our knowledge operator. Replacing Bp with $\neg K \neg Kp$ in (CB) also reveals that the following additional principle must be valid in our resulting logic of pure knowledge (the K-fragment of S).

$$\neg K \neg K p \rightarrow K \neg K \neg p$$

This is the K version of the **G** axiom, and it can be shown that the result of adding this as an axiom to **S4** is the knowledge fragment of S.

Proposition 2.1.10. **S4G** *(=***S4.2***) is the epistemic fragment of* S.

Given the above two results then we can now determine what the logic of the doxastic fragment of S is – namely it will be the $\{\neg K \neg K, \rightarrow, \neg\}$-fragment of **S4G**. That is to say, the unique normal modal logic S such that $A \in \mathsf{S}$ if and only if $\tau(A) \in$ **S4G** – where $\tau(B\varphi) = \neg K \neg K \tau(\varphi)$. It is easy to see that the doxastic fragment of S is going to be at least as strong as the logic **KD45** – **4** and **5** following from (PI) and (NI) respectively using (KB). Thus, as the doxastic fragment of S is faithfully embedded into **S4G** by τ we have the following.

Theorem 2.1.11. $A \in$ **KD45** *if and only if* $\tau(A) \in$ **S4G** *where* $\tau(B\varphi) = \neg K \neg K \tau(\varphi)$.

[7] That is to say, in addition to the above principles we have $K(p \rightarrow q) \rightarrow (Kp \rightarrow Kq)$, $Kp \rightarrow p$ and $Kp \rightarrow KKp$ as theorems.

One of the most famous combined principles of knowledge and belief is the idea that knowledge is equivalent to true belief. It is generally thought that this concept of knowledge over generates, in the sense that it counts as instances of knowledge things which the agent does not know. Consequently we can think of the epistemic fragment of S extended by the following principle as giving us a hard upper bound on what our formal theory of knowledge could be.

$$(KTB) \quad K\varphi \leftrightarrow B\varphi \wedge \varphi.$$

The 'only if' direction of the above biconditional is already provable in S by (KB), **T** and the definition of B above. Thus, we can provide a more economical axiomatization of this logic of knowledge as the epistemic fragment of S by $B\varphi \wedge \varphi \to K\varphi$. By the definition of B given above this forces our epistemic fragment to prove the following formula.

$$.4: \quad \varphi \wedge \neg K \neg K\varphi \to K\varphi$$

This tells us that the logic of knowledge as true belief, where belief is taken to be subjective certainty is none other than the normal modal logic **S4.4**. Thus, for reasons outlined above, we can think of **S4.4** as being a maximal epistemic logic compatible with taking belief as subjective certainty – i.e. no proper extension of **S4.4**, construed as an epistemic logic, is compatible with belief being subjective certainty.[8] Proposition 2.1.10 by contrast tells us that the smallest logic of knowledge where belief is taken as subjective certainty is the normal modal logic **S4.2**. Consequently we are left with a partial characterization of the admissible epistemic logics if we take belief as subjective certainty – namely those modal logics extending **S4.2** which do not extend **S4.4**.[9]

[8] This result is rendered in more detail in Theorem 4.2.18

[9] This much more cautious way of phrasing things here is due to comments of S. Kuhn. Previously the author had written that the above results show that **S4.4** is the strongest epistemic logic compatible with this construal of belief, and that this meant that all admissible epistemic logics were in the interval between **S4.2** and **S4.4**. This, while being how these results are reported in Lenzen [1978], does not clearly follow from the above.

2.1.1.2 Alethic Modal Logic: Metaphysical Necessity and Actuality

The most commonly accepted modal logic of metaphysical necessity is the normal modal logic **S5**.[10] From a semantic point of view, **S5** corresponds to the notion of metaphysical necessity as truth in all possible worlds. The most prominent opponent to this orthodox view is Nathan Salmon, who in Salmon [1989] has argued that there are counterexamples to the **S5** principles for metaphysical necessity arising over the constitution of artefacts. Salmon argues that those who find **S5** to be a compelling logic of metaphysical necessity are confusing necessity with actual necessity – what is necessary according to the actual world. Let $\tau_{\mathsf{A}\Box}$ by the translation which replaces all occurrences of $\Box$ within a formula it is applied to with $\mathsf{A}\Box$. Then, in order for Salmon to be correct one thing we would have to be able to show is that the set of all formulas $\tau_{\mathsf{A}\Box}(A)$ is some system of modal logic S which proves neither **4** or **5** is exactly **S5**. What we will now show is that there is such a logic, before going on to use this argument to assess Salmon's argument.

Let **KT@** be the normal bimodal logic containing two operators $\Box$ and A axiomatized by the following axioms, with modus ponens, uniform substitution, and necessitation for $\Box$ and A ('actualization') as its sole rules.

$$\begin{array}{rcl}
\mathbf{K} & : & \Box(p \to q) \to (\Box p \to \Box q) \\
\mathbf{T} & : & \Box p \to p \\
\mathbf{K}_{\mathsf{A}} & : & \mathsf{A}(p \to q) \to (\mathsf{A}p \to \mathsf{A}q) \\
\mathbf{A1} & : & \mathsf{A}\neg p \leftrightarrow \neg \mathsf{A}p \\
4_{\mathsf{A}} & : & \mathsf{A}p \to \mathsf{AA}p \\
\mathbf{A2} & : & \mathsf{A}p \to \Box \mathsf{A}p
\end{array}$$

As stated in Williamson [1998], **KT@** corresponds to an interpretation of A on which $\mathsf{A}p$ is true at a world w just in case p is true at a fixed world w^*. The theorems of **KT@** are those which are valid on all frames $\langle W, R, w^* \rangle$

[10] The following example is taken from Williamson [1998].

when A is interpreted in this way. That is to say, in the terminology of Humberstone [2004], **KT@** is the class of formulas which are *generally valid* on the class of all reflexive frames $\langle W, R, w^* \rangle$. By contrast **KT@S**, the smallest modal logic extending **KT@** containing all instances of the formula $\mathsf{A}p \to p$, is the class of all formulas which are *real-world valid* on all reflexive frames $\langle W, R, w^* \rangle$ – i.e. the class of all formulas A such that, for all reflexive models we have $\langle W, R, w^*, V \rangle \models_{w*} A$.

What we will now show is that $\{A | \tau_{\mathsf{A}\Box}(A) \in \mathbf{KT@S}\} = \mathbf{S5}$ – i.e. that $\tau_{\mathsf{A}\Box}$ faithfully embeds **S5** into **KT@S**.

Proposition 2.1.12. *For all formulas A we have the following.*

$$A \in \mathbf{S5} \textit{ only if } \tau_{\mathsf{A}\Box}(\Box A) \in \mathbf{KT@}.$$

Proposition 2.1.13.

$$\tau_{\mathsf{A}\Box}(\Box A) \to \tau_{\mathsf{A}\Box}(A) \in \mathbf{KT@S}.$$

Proof. As **KT@S** contains all formulas of the form $\mathsf{A}p \to p$, we know that $\mathsf{A}\Box\tau_{\mathsf{A}\Box}(A) \to \Box\tau_{\mathsf{A}\Box}(A) \in \mathbf{KT@S}$, and hence by **T** that $\mathsf{A}\Box\tau_{\mathsf{A}\Box}(A) \to \tau_{\mathsf{A}\Box}(A) \in$ **KT@S**. □

Theorem 2.1.14 (Williamson [1998])**.** *For all formulas A we have the following.*

$$A \in \mathbf{S5} \textit{ if and only if } \tau_{\mathsf{A}\Box}(A) \in \mathbf{KT@S}.$$

Proof. For the 'only if' direction suppose that $A \in \mathbf{S5}$. Then by Proposition 2.1.12 we know that $\tau_{\mathsf{A}\Box}(\Box A) \in \mathbf{KT@}$, and thus by Proposition 2.1.13 that $\tau_{\mathsf{A}\Box}(A) \in \mathbf{KT@S}$.

For the 'if' direction suppose that $A \notin \mathbf{S5}$. By the soundness and completeness of **S5** w.r.t the class of all universal models we know that there is a universal modal $\mathcal{M} = \langle W, R, V \rangle$ and a point $x \in W$ such that $\mathcal{M} \not\models_x A$. Singling out x as the actual world we get a model for **KT@S** in which A is still false and in which $A \leftrightarrow \tau_{\mathsf{A}\Box}(A)$ is valid. Hence we can conclude that $\langle W, R, x, V \rangle \not\models_x \tau_{\mathsf{A}\Box}(A)$ and thus that $\tau_{\mathsf{A}\Box}(A) \notin \mathbf{KT@S}$.

□

We cannot show the above result with **KT@** replacing **KT@S** – the translation failing in the 'only if' direction – $\mathsf{A}\Box p \to p$ not being a theorem of **KT@**, this being the $\tau_{\mathsf{A}\Box}$-translation of the **T** axiom. As the theorems of **KT@S** are those which are 'real world valid' on reflexive frames, it appears as if the above argument concerning the confusion of necessity and actual necessity relies on us thinking of our candidate metaphysical modal logic(s) as characterizing the behaviour of metaphysical necessity according to the actual world. But surely when we are talking about **S5** being the logic of metaphysical necessity we are claiming that it characterizes metaphysical necessity in a less world-relative manner. That is to say, the success of Salmon's argument seems to rest on whether one thinks that we should characterize metaphysical necessity in terms of real-world valid, or generally valid formulas. If instead we were to characterize metaphysical necessity in terms of the generally valid formulas, then we would end up with **KT@** as our preferred source logic, and the embedding would no longer be faithful. What Salmon's argument seems to suggest, then, is that people who think that **S5** is the logic of metaphysical necessity had better be characterizing their logics in terms of general validity.

III

The Range of Translations

Given the definition of what it is for a translation τ for faithfully embed a logic S into a logic S' there are three different elements which we can vary – we can (i) vary the translation τ, (ii) vary our source logic S or (iii) vary our target logic S'. Indeed, it is very natural to wonder whether there is anything interesting to be said about each of these options, both in general terms and for specific choices of translation, source and target logic.

Option (i), where we vary our translation and keep our source and target logic fixed, asks us for which translations τ can we faithfully embed S into S'. For specific logics we can have some interesting things to say about this (we will find an example of this phenomenon in our discussion of the logic of action), but in the general case the only time we can say something interesting about this situation is when the answer is 'none'. For example we can show that for no modal-to-modal translation τ can we faithfully embed $\mathbf{K}$ into $\mathbf{KT}$.

Theorem 3.0.15 (Humberstone [2005a]). *No modal-to-modal translation faithfully embeds* $\mathbf{K}$ *into* $\mathbf{KT}$.

Proof. Suppose, for a reductio, that τ faithfully embeds $\mathbf{K}$ into $\mathbf{KT}$. Then as $\Diamond\top \vee \Box\bot \in \mathbf{K}$ it follows that $\tau(\Diamond\top \vee \Box\bot) \in \mathbf{KT}$. As τ is a modal-to-modal translation this means that $\tau(\Diamond\top) \vee \tau(\Box\bot) \in \mathbf{KT}$. As $\mathbf{KT}$ is Halldén complete[1] then it follows that either $\tau(\Diamond\top) \in \mathbf{KT}$ or $\tau(\Box\bot) \in \mathbf{KT}$. But neither of $\Diamond\top$ nor $\Box\bot$ are $\mathbf{K}$-theorems, and hence by reductio there is no such translation τ. $\Box$

The above Theorem is an instance of a more general result that for any translation for which $\tau(A \vee B) = \tau(A) \vee \tau(B)$ that τ faithfully embeds S into S' only if whenever S' is Halldén complete, S is too.

Option (ii), where we vary our source logic and keep our translation and target logic fixed, asks us which source logics S can be faithfully embedded into S' by τ. This question has a general answer, albeit a not very interesting one. We know that, for all choices of target logic S' and translation τ that if there is a source logic S which can be faithfully embedded into S' by τ, then there is only one such logic – namely the logic $\{A|\tau(A) \in \mathsf{S}'\}$. If we are dealing with compositional and variable-fixed translations then it is clear that this set will be a logic whenever S' is – as $\sigma'(A)$ will be in the set whenever $\sigma(\tau(A))$ is in S', where $\sigma(p_i) = \tau(\sigma'(p_i))$, this fact following from Theorem 2.0.20. So, for example, we can show that if S' is a modal logic and τ a modal-to-modal translation then the set $\{A|\tau(A) \in \mathsf{S}'\}$ will always be a logic, but in general we are not assured of such. Moreover, here we are not assured that the logic in question will be a modal logic – as we are not assured that all the classical tautologies will be in the set. So we can see that the more interesting problem here is determining properties of the logic $\{A|\tau(A) \in \mathsf{S}'\}$.

Option (iii), where we keep our source logic and translation fixed while varying our target logic, asks us what logics our source logic S can be faithfully embedded into by τ. This option, it turns out, is by far the most interesting in the general case – as there are often a set of logics S' which

[1] A modal logic S is *Halldén complete* iff whenever $A \vee B \in \mathsf{S}$, where A and B do not share a propositional variable, either $A \in \mathsf{S}$ or $B \in \mathsf{S}$.

S can be faithfully embedded into by τ. Not only this though, but we can show that this set is endowed with certain algebraic properties under the usual ordering relation. What we will look at in this chapter is what we can say about this option for abstracting our definition of what it is for one logic to be faithfully embedded into another, focusing in particular on the structures formed by such sets of logics under the partial ordering $\subseteq$.

Let us begin by establishing some terminology. Consider the following relation between logics – for a given choice of translation τ.

$$\mathsf{S}' \equiv_\tau \mathsf{S}'' \quad =_{Df} \quad \text{for all formulas } A\text{: } \tau(A) \in \mathsf{S}' \iff \tau(A) \in \mathsf{S}''. \tag{3.1}$$

Let us say that two logics S' and S'' are τ-*equivalent* whenever $\mathsf{S} \equiv_\tau \mathsf{S}'$. It is easy to see that the set of logics into which a given logic S can be faithfully embedded by a translation τ is the τ-equivalence class of logics Γ, such that $\mathsf{S}' \in \Gamma$ only if τ faithfully embeds S in S'.Let us denote this set of logics $Ran(\tau, \mathsf{S})$ – calling it the *range of* τ *for* S. Additionally, let us denote by $NRan(\tau, \mathsf{S})$ the set of *normal* modal logics in $Ran(\tau, \mathsf{S})$. The nomenclature here is chosen to bring to mind the idea of the range of a function – the set of output values produced by the function, in this case being the set of (target) logics into which our translation faithfully embeds our source logic. The similarity is rather coarse, but suggestive nonetheless.

The main question with which we will be concerned in this chapter is what can be said about the range of a translation τ for a given source logic S. In particular our focus will be on determining what can be said about the structure formed by the set $Ran(\tau, \mathsf{S})$ ($NRan(\tau, \mathsf{S})$) and the usual ordering $\subseteq$. In doing this it will be useful to be clear on some terminology involving sets and relations on them. Let X be a nonempty set and $>$ a partial order on X. Then an element $x \in X$ is *minimal in* X if for all $y \in X$, if $x \geq y$ then $x = y$. An element is $x \in X$ is *the minimum of* X if for all $y \in X$ $x \leq y$. Similarly, an element $x \in X$ is *maximal in* X if for all $y \in X$, if $x \leq y$ then $x = y$, and is *the maximum of* X if for all $y \in X$, $y \leq x$. Note that all minimum elements are minimal elements, and all maximum elements are maximal, but not conversely.

Results of this nature exist most prominently in the literature concerning the modal companions of intermediate logics. The objects of study here are the translations which faithfully embed IL (Intuitionistic Logic) into the normal modal logic **S4**. In particular consider the translation T due to Gödel.[2]

$$\begin{aligned} T(p_i) &= \Box p_i \\ T(A \vee B) &= \Box(T(A) \vee T(B)) \\ T(A \wedge B) &= \Box(T(A) \wedge T(B)) \\ T(A \to B) &= \Box(T(A) \to T(B)) \\ T(\neg A) &= \neg\Diamond T(A). \end{aligned}$$

From this literature it is known that the minimal normal modal logic into which IL can be faithfully embedded by T is **S4**, and the maximal such logic is **S4Grz** – the normal extension of **S4** by the formula **Grz** (= $\Box(\Box(p \to \Box p) \to p) \to p)$).

That is to say, the normal modal logics into which IL can be faithfully embedded by T are all those logics in the interval $[\mathbf{S4}, \mathbf{S4Grz}]$. This result is a paradigm case of the kinds of result which we will be primarily concerned with in this chapter. Results of this nature in the modal logic literature are rare. In Shavrukov [1991] we are told what the maximal logic extending **GL** into which the translation $\tau_{\boxdot}$ faithfully embeds **Grz** is. When combined with the results in Litak [2007] this allows us to start to get a very vague picture of what the structure of $NRan(\tau_{\boxdot}, \mathbf{Grz})$ might be – which we will go into later this chapter. The most complete example of this kind of reasoning present in the literature is Goris [2007], where we

[2]For more information on Intermediate Logics and translational embeddings between them and extensions of **S4** (known in this literature as 'Modal Companions') the reader should consult the survey article Chagrov & Zakharyashchev [1992]. More recent results in this field can also be seen in Muravitsky [2006].

are shown what the structure of $NRan(\tau_{\boxdot},\mathbf{S5})$ and $NRan(\tau_{\boxdot},\mathbf{S4.4})$ are.

One obvious question to ask when presented with the characteristic function for a set is whether we can determine conditions under which that set will be nonempty. In this case the characteristic function we have been presented with is "being a logic into which S can be faithfully embedded by τ", this being the characteristic function for the set $Ran(\tau,\mathsf{S})$. As such we want to know whether we can give some condition or conditions under which we can be assured that $Ran(\tau,\mathsf{S})$ is non-empty. The first thing to notice in doing this is that it is not the case that the range of τ for S will always be non-empty. To see this consider the range of $\tau_{\boxdot}$ for $\mathbf{K}$. As every modal logic proves the formula $(p \wedge \Box p) \to p$ (this being a substitution instance of the classical theorem $(p \wedge q) \to p$), and as this formula is $\tau_{\boxdot}(\Box p \to p)$ in order for $Ran(\tau_{\boxdot},\mathbf{K})$ to be nonempty '$\Box p \to p$' would have to be a $\mathbf{K}$-theorem. As it is not we can conclude that $Ran(\tau_{\boxdot},\mathbf{K}) = \emptyset$. Moreover, this observation will hold for any extension of $\mathbf{K}$ which does not include $\mathbf{T}$ amongst its theorems.

Are there any conditions under which the range of a translation will be nonempty? One good place to start is in considering the smallest modal logic containing $\tau(A)$ for every S-theorem A, which is to say the logic $\mathbf{L} + \tau(\mathsf{S})$. A necessary and sufficient condition for $Ran(\tau,\mathsf{S})$ to be non-empty is that τ faithfully embeds S into $\mathbf{L} + \tau(\mathsf{S})$. What this means is that the smallest modal logic containing $\tau(A)$ for every S-theorem A, contains no formula $\tau(B)$ for which $B \notin \mathsf{S}$.

For reasons of expository simplicity, for the remainder of this chapter will be dealing with sets $Ran(\tau,\mathsf{S})$ which are non-empty.

3.1 The Minimum Logic

Suppose that we have established that $Ran(\tau,\mathsf{S})$ is non-empty. An obvious thing for us to ask is whether it contains a minimum element. If there was to be such a least member a good initial candidate would be the logic $\mathbf{L} +$

$\tau(\mathsf{S})$ mentioned above. In fact, as we will now show, this is the minimum logic within $R(\tau, \mathsf{S})$.

Theorem 3.1.1. *For all modal logics* S', *if* S *is faithfully embedded into* S' *by* τ, *then* $\mathbf{L} + \tau(\mathsf{S}) \subseteq \mathsf{S}'$.

Proof. Suppose that τ embeds S into a logic S' faithfully. Then S' proves all of the formulas in $\tau(\mathsf{S})$. Additionally, as S' is a modal logic it will also prove all the formulas in $\mathbf{L}+\tau(\mathsf{S})$, and thus we can see that $\mathbf{L}+\tau(\mathsf{S}) \subseteq \mathsf{S}'$. $\square$

Thus we can see that, whenever the range of a translation is non-empty that it has a least member. What we would really like though is a set of axioms and rules which characterize this least member. It is reasonably clear from the construction of $\mathbf{L} + \tau(\mathsf{S})$ that, if S is a modal logic given in terms of some set of axioms $A_1, \ldots, A_n$ along with the rules of Modus Ponens and Uniform substitution then $\mathbf{L}+\tau(\mathsf{S})$ can be given in terms of the axioms $\tau(A_1), \ldots, \tau(A_n)$ along with the rules of Modus Ponens and Uniform Substitution.

What about $NRan(\tau, \mathsf{S})$? As above it is fairly easy to see that $NRan(\tau, \mathsf{S})$ will be nonempty iff it contains the logic $\mathbf{K} \oplus \tau\mathsf{S}$ – the minimum normal modal logic extending $\tau\mathsf{S}$. This logic can be given axiomatics similar to that of $\mathbf{L} + \tau(\mathsf{S})$, except that we will now also have the additional axiom $\mathbf{K}$ as well as the rule of necessitation.

3.1.1 Example: The minimal logic in $NRan(\tau_{\Box\Box}, \mathbf{KD}_c)$

Throughout this chapter we will investigate, as a working example, the structure of $NRan(\tau_{\Box\Box}, \mathbf{KD}_c)$ – where $\tau_{\Box\Box}$ is the modal translation such that $\tau_{\Box\Box}(\Box A) = \Box\Box\tau_{\Box\Box}(A)$. We will begin here by determining what the minimal logic in this range is. To that end consider the following formula.

$$\mathbf{KD}_c^2: \quad \Diamond\Diamond p \to \Box\Box p.$$

It is easy to see that $\mathbf{KD}_c^2$ is determined by the class of all frames $\langle W, R\rangle$ where R^2 partially functional. What is perhaps more interesting is that all of the generated frames for $\mathbf{KD}_c^2$ are very easily describable. Given a frame generated by a point w – $\mathfrak{F} = \langle W, R\rangle$ – and a set of points $C_n = \{x, y_1, \ldots, y_n\}$ such that W and C_n are disjoint, let us denote by $f_{C_n}(\mathfrak{F})$ the following frame.

- $W_{C_n} := W \cup C_n$
- $R_{C_n} := R \cup \{\langle x, y_i\rangle, \langle y_i, w\rangle | 1 \leq i \leq n\}$.

What we can quite easily show is that all the generated frames for $\mathbf{KD}_c^2$ are either generated frames for $\mathbf{KD}_c$, or $f_{C_n}(\mathfrak{F})$ for some generated frame for $\mathbf{KD}_c$.

Definition 3.1.2. Given a model $\mathcal{M} = \langle W, R, V\rangle$ let us define the set of points $I = \{I(x, y) | \langle x, y\rangle \in R\}$. Construct the model $\mathcal{M}^* = \langle W^*, R^*, V\rangle$ as follows.

- $W^* := W \cup I$
- $R^* := \{\langle x, I(x, y)\rangle | \langle x, y\rangle \in R\} \cup \{\langle I(x, y), y\rangle | \langle x, y\rangle \in R\}$.

The above model construction takes a model, and puts intermediary points between R-related points. Note the similarity between this construction, used in Pelletier & Urquhart [2008] and Kuhn [2004], and the related construction seen in Chapter 2 inspired by a similar construction in Humberstone [2006].

Lemma 3.1.3. *Let $\mathcal{M} = \langle W, R, V\rangle$ be a Kripke model. Then for all formulas A and points $x \in W$ we have the following.*

$$\mathcal{M} \models_x A \textit{ if and only if } \mathcal{M}^* \models_x \tau_{\Box\Box}(A).$$

Proof. By induction upon the complexity of A, the only case of interest being that in the inductive step where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that $\mathcal{M} \models_x \Box B$. Then for all $y \in R(x)$ we know that $\mathcal{M} \models_y B$. By the inductive hypothesis it follows that $\mathcal{M}^* \models_y \tau_{\Box\Box}(B)$ for all such points y. As Rxy we know that $RxI(x,y)$ and $RI(x,y),y$ and thus that $\mathcal{M}^* \models_x \Box\Box\tau_{\Box\Box}(B)$ as desired.

For the 'if' direction suppose that $\mathcal{M}^* \models_x \Box\Box\tau_{\Box\Box}(B)$. Then for all points y such that R^*xy we have that $\mathcal{M}^* \models_y \Box\tau_{\Box\Box}(B)$, and thus for all points z we know that $\mathcal{M}^* \models_z \tau_{\Box\Box}(B)$. By the inductive hypothesis we know that $\mathcal{M} \models_z B$. By the construction of R^* we know that all such points y are of the form $I(x,z)$ for some points $z \in W$ – and thus that Rxz for all such points z, from which it follows that $\mathcal{M} \models_x \Box B$ as desired. □

Theorem 3.1.4. *For all formulas A we have the following.*

$$A \in \mathbf{KD}_c \textit{ if and only if } \tau_{\Box\Box}(A) \in \mathbf{KD}_c^2.$$

Proof. The 'only if' direction proceeds by induction upon the length of derivations of A in $\mathbf{KD}_c$. The only case of interest being the basis case where A is an instance of $\mathbf{D}_c$. In this, we know that $\tau_{\Box\Box}(\mathbf{D}_c)$ is provable in $\mathbf{KD}_c^2$, and thus – by taking the appropriate substitution instance of $\mathbf{D}_c^2$ that $\tau_{\Box\Box}(A) \in \mathbf{KD}_c^2$.

For the 'if' direction suppose that $A \notin \mathbf{KD}_c$. Then there is a partially functional model $\mathcal{M} = \langle W, R, V\rangle$ and a point $x \in W$ such that $\mathcal{M} \not\models_x A$. By Lemma 3.1.3 we know that $\mathcal{M}^* \not\models_x \tau_{\Box\Box}(A)$ and thus, as this is a model for $\mathbf{KD}_c^2$ that $\tau_{\Box\Box}(A) \notin \mathbf{KD}_c^2$. □

All we have done so far is show that $\mathbf{KD}_c^2 \in NRan(\tau_{\Box\Box}, \mathbf{KD}_c)$. What we will now proceed to show is that it is the minimum such logic.

Theorem 3.1.5. *$\mathbf{KD}_c^2$ is the minimum logic in $NRan(\tau_{\Box\Box}, \mathbf{KD}_c)$.*

Proof. Suppose that $\mathsf{S} \in NRan(\tau_{\Box\Box}, \mathbf{KD}_c)$. Then we know that S is normal, and also proves $\tau_{\Box\Box}(\Diamond p \to \Box p) = \Diamond\Diamond p \to \Box\Box p$. Thus, as $\mathbf{KD}_c^2$ is the smallest normal modal logic which proves $\Diamond\Diamond p \to \Box\Box p$ we can see that $\mathsf{S} \supseteq \mathbf{KD}_c^2$, and the result follows. □

3.2 Maximal Logics

While the problem of determining what the minimal logics within the range of a translation is for a given source logic is relatively simple – there in fact always being a minimum element whenever $Ran(\tau, \mathsf{S})$ is non-empty, the problem of finding maximal logics is somewhat harder – further complicated by the fact that there is no guarantee that there will be only one such logic. What we will show in this section though are some results and general approaches which can be helpful when trying to determine the maximal logics within the range of a translation. It will be useful in what follows to have some terminology to talk about the maximal logics within the range of a translation. For this reason let us introduce the following terminology. Let $max(Ran(\tau, \mathsf{S}))$ (resp. $max(NRan(\tau, \mathsf{S}))$) be the smallest set $\Delta \subseteq Ran(\tau, \mathsf{S})$ (resp. $\Delta \subseteq NRan(\tau, \mathsf{S})$) such that if $\mathsf{S}' \in Ran(\tau, \mathsf{S})$ (resp. $\mathsf{S}' \in NRan(\tau, \mathsf{S})$) then $\mathsf{S}' \subseteq \mathsf{S}''$ for some logic $\mathsf{S}'' \in \Delta$.

We will begin by giving a condition which, when satisfied by a logic S for a translation τ, is sufficient for S being maximal in $Ran(\tau, \mathsf{S})$.

We will begin by giving a condition upon our source logic which is sufficient for it being maximal. This result applies not only to modal-to-modal translations, but also to any translation where the source and target logics share the same propositional language.

Theorem 3.2.1. *Suppose that τ fulfils the following condition:*

$$\vdash_{\mathsf{S}} \tau(A) \leftrightarrow A.$$

Then S is a maximal logic into which S can be faithfully embedded via τ.

Proof. First we note that it is trivial to show that any translation fulfilling the condition above can faithfully embed S into S.

To show that S is maximal, suppose for a contradiction that there is a logic S' such that $\mathsf{S}' \supsetneq \mathsf{S}$ such that S is faithfully embedded into S' by τ. As $\mathsf{S}' \supsetneq \mathsf{S}$ there is some formula $A \in \mathsf{S}'$ such that $A \notin \mathsf{S}$. By the above inset condition then we know that $\tau(A) \in \mathsf{S}'$, and thus by the faithfulness of the translation that $A \in \mathsf{S}$, giving us a contradiction. $\square$

This result allows us to conclude the following.

Theorem 3.2.2. **KT** *is maximal in* $Ran(\tau_{\boxdot}, \mathbf{KT})$.

Proof. Follows by Theorem 3.2.1, and the fact that **KT** is congruential and proves $(\Box p \wedge p) \leftrightarrow \Box p$. $\square$

We will have more to say about the above result (that **KT** is maximal in $NRan(\tau_{\boxdot}, \mathbf{KT})$) at the end of the next chapter, where we consider the question of whether **KT** is in fact the maximum logic in this range.

3.2.1 Example: $\mathbf{KD}_c$ is maximal in $NRan(\tau_{\Box\Box}, \mathbf{KD}_c)$

What we will show here is that $\mathbf{KD}_c$ is maximal within $NRan(\tau_{\Box\Box}, \mathbf{KD}_c)$. To do this we will show that every proper normal extension of $\mathbf{KD}_c$ proves the $\tau_{\Box\Box}$-translation of some formula A such that A is not $\mathbf{KD}_c$-provable. First we will require some results from Segerberg [1986] concerning the normal extensions of $\mathbf{KD}_c$. Let $\mathbf{C}_n$ be the following formula, named for Brian Chellas.

$$\mathbf{C}_n: \quad \Box^n \Diamond \top.$$

Proposition 3.2.3. *For all* $n \in Nat$, $\tau_{\Box\Box}(\mathbf{C}_n) \in \mathbf{KD}_c\mathbf{C}_n$.

Proof. First note that $(\Diamond\top \wedge \Box\Diamond\top) \to \Diamond\Diamond\top \in \mathbf{K}$, and hence by repeated applications of **RR** we have that $(\Box^{2n}\Diamond\top \wedge \Box^{2n}\Box\Diamond\top) \to \Box^{2n}\Diamond\Diamond\top \in \mathbf{K}$. By repeated applications of the rule of necessitation applied to $\mathbf{C}_n$ we have that $\Box^{2n}\Diamond\top$ and $\Box^{2n}\Box\Diamond\top$ are both theorems of $\mathbf{KD}_c\mathbf{C}_n$, and thus by Modus Ponens that $\Box^{2n}\Diamond\Diamond\top \in \mathbf{KD}_c\mathbf{C}_n$. As this is just the $\tau_{\Box\Box}$-translation of $\mathbf{C}_n$ the result follows. $\square$

Proposition 3.2.4. *For all* $n \in Nat$, $\mathbf{KD}_c\mathbf{C}_n \notin NRan(\tau_{\Box\Box}, \mathbf{KD}_c)$.

Proof. Follows by Proposition 3.2.3 and the fact that $\mathbf{C}_n$ is not a theorem of $\mathbf{KD}_c$. $\square$

Proposition 3.2.5. *Suppose that* S *is a proper normal extension of* $\mathbf{KD}_c$*. Then* $\mathbf{KD}_c\mathbf{C}_n \subseteq \mathsf{S}$ *for some* $n \in Nat$

Proof. By Lemma 2.7 and 2.8 of Segerberg [1986]. $\square$

Theorem 3.2.6. $\mathbf{KD}_c$ *is maximal within* $NRan(\tau_{\Box\Box}, \mathbf{KD}_c)$.

Proof. By Proposition 3.2.5 we know that every logic $\mathbf{KD}_c \subsetneq \mathsf{S}$ is an extension of one of the logics $\mathbf{KD}_c\mathbf{C}_n$ for some $n \in Nat$. By Proposition 3.2.4 it follows that $\mathbf{KD}_c\mathbf{C}_n$ proves $\tau_{\Box\Box}(A)$ for some formula $A \notin \mathbf{KD}_c$. Thus, as $\mathbf{KD}_c\mathbf{C}_n \subseteq \mathsf{S}$ it follows that $\tau_{\Box\Box}(A) \in \mathsf{S}$ and hence $\mathsf{S} \notin NRan(\tau_{\Box\Box}, \mathbf{KD}_c)$. $\square$

What we have shown so far is that $\mathbf{KD}_c$ is one of the maximal logics in $NRan(\tau_{\Box\Box}, \mathbf{KD}_c)$, leaving it open as to whether there are any other maximal logics within the range. What we will do now is show that if there are any such logics, that they are not determined by a class of Kripke frames – this being at least partial inductive evidence that $\mathbf{KD}_c$ is the maximum logic within the range.

Let $\beta_n = \langle W_n, R_n \rangle$ be the following frame taken from Segerberg [1986, p.506].

- $W_n := \{i | i < n\}$,
- $R_n := \{\langle i, i+1 \rangle | i < n-1\}$.

Lemma 3.2.7. *Suppose that* $\mathbf{KD}_c^2 \subseteq \mathsf{S}$ *is Kripke-complete. Then if* $\beta_n \notin Fr(\mathsf{S})$ *then* $\neg(\Diamond^{n+2}\Box\bot) \in \mathsf{S}$.

Proof. Suppose for a contradiction that $\beta_n \notin Fr(\mathsf{S})$ and that $\neg(\Diamond^{n+2}\Box\bot) \notin \mathsf{S}$. Then there is a model $\mathcal{M} = \langle W, R, V \rangle$ and a point $x \in W$ such that $\mathcal{M} \models_x \Diamond^{n+2}\Box\bot$. So there is are points $y, z \in W$ such that Rxy and Ryz and $\mathcal{M} \models_z \Diamond^n\Box\bot$. It is easy to see that the frame of the model generated from $\mathcal{M}$ at z is partially functional, and hence is simply β_n – contradicting out hypothesis. $\square$

Theorem 3.2.8. *For all Kripke-complete normal modal logics* S *we have the following.*

$$\mathsf{S} \in NRan(\tau_{\Box\Box}, \mathbf{KD}_c) \text{ if and only if } \mathsf{S} \subseteq \mathbf{KD}_c.$$

Proof. For the 'if' direction suppose that $\mathsf{S} \nsubseteq \mathbf{KD}_c$. So we know that for some point generated finite frame $\mathfrak{F} \in Fr(\mathbf{KD}_c)$ that $\mathfrak{F} \notin Fr(\mathsf{S})$. As all the point generated finite frames for $\mathbf{KD}_c$ are of the form β_n for some $n \in Nat$ it follows that $\beta_n \notin Fr(\mathsf{S})$ for some $n \in Nat$. Thus by the above Lemma we know that $\neg(\Diamond^2\Diamond^{n-1}\Box\bot) \in \mathsf{S}$ – that is $\Box^{n+1}\Diamond\top \in \mathsf{S}$. Thus it follows that $\mathsf{S} \supseteq \mathbf{KD}_c\mathbf{C}_{n+1}$, and thus by Proposition 3.2.3 that $\tau_{\Box\Box}(\mathbf{C}_{n+1}) \in \mathsf{S}$. As $\mathbf{C}_{n+1} \notin \mathbf{KD}_c$ it follows then that $\mathsf{S} \notin NRan(\tau_{\Box\Box}, \mathbf{KD}_c)$.

The 'only if' direction is routine. □

3.3 The Structure of the Range of a Translation

Given that we have now shown how to determine (in at least some limited cases) what the minimal and maximal logics into which a given logic S can be faithfully embedded by a translation τ, we will now examine some of the broader structural properties of the range of a translation.

The first thing which we might want to know is what kind of structure the range of a translation forms under the obvious partial ordering $\subseteq$. Given that the set of all modal logics forms a lattice under $\subseteq$ this would be an obvious place to start.

Proposition 3.3.1. $Ran(\tau, \mathsf{S})$ *is closed under intersections.*

Proof. Suppose that S' and S'' are in $R(\tau, \mathsf{S})$, with a view to showing that their intersection $\mathsf{S}' \cap \mathsf{S}''$ is in $R(\tau, \mathsf{S})$ also. Firstly if $A \in \mathsf{S}$ then $\tau(A) \in \mathsf{S}'$ and $\tau(A) \in \mathsf{S}''$ and consequently $\tau(A) \in \mathsf{S}' \cap \mathsf{S}''$. If $\tau(A) \in \mathsf{S}' \cap \mathsf{S}''$ then $\tau(A) \in \mathsf{S}'$ and $\tau(A) \in \mathsf{S}''$ and thus $A \in \mathsf{S}$ as desired. □

Corollary 3.3.2. $NRan(\tau, \mathsf{S})$ *is closed under intersections.*

Thus it is very easy to see that the meet of any two logics within the range of a translation is also within the range of that translation. The case of joins though is far more problematic due to the fact that we can have pairwise incomparable maximal modal logics with the range of a translation – the following constituting a concrete counterexample to the range of a translation being closed under join, where the join of two modal logics S and S' is the smallest modal logic S'' such that $\mathsf{S} \subseteq \mathsf{S}''$ and $\mathsf{S}' \subseteq \mathsf{S}''$.

Example 3.3.3. Consider the join of the two logics **KD45** and **S4.4** in the range of $\tau_{\Diamond\Box}$ for **KD45**. This logic, which turns out to be the normal modal logic **S5**,[3] contains all of the theorems of **KD45** and **S4.4** and is closed under Modus Ponens. Consequently, as $\Diamond\Box p \to \Box p \in \mathbf{KD45}$ and $\Box p \to p \in \mathbf{S4.4}$ it is clear that $\Diamond\Box p \to p$ will be in their join. But this is the $\tau_{\Diamond\Box}$-translation of the **KD45**-unprovable formula $\Box p \to p$, and thus this logic cannot be in $Ran(\tau_{\Diamond\Box}, \mathbf{KD45})$.

This does not mean that the range of a translation can never be closed under joins – what we are trying to determine here is what can be said about the structure of $Ran(\tau, \mathsf{S})$ and $NRan(\tau, \mathsf{S})$ in general. As it happens one of our examples ($NRan(\tau_{\boxdot}, \mathbf{KT!})$) forms an interval – and thus is closed under both meets and joins. In general though we can conclude the following.

Proposition 3.3.4. *$Ran(\tau, \mathsf{S})$ forms a bounded meet semi-lattice with $0 = \mathbf{L} + \tau\mathsf{S}$.*

Proposition 3.3.5. *$NRan(\tau, \mathsf{S})$ forms a bounded meet semi-lattice with $0 = \mathbf{K} \oplus \tau\mathsf{S}$.*

An additional structural property, which is of practical use, is the following convexity result.

[3]This is an instance where even when we just take the smallest modal logic which contains all the theorems of two logics, we nonetheless end up with a normal modal logic.

Theorem 3.3.6 (Convexity). *For all logics* S' *such that* $\mathsf{S}_0 \subseteq \mathsf{S}' \subseteq \mathsf{S}_1$ *– where* S *can be faithfully embedded into both* S_0 *and* S_1 *by* τ *– we have that:*

$$A \in \mathsf{S} \textit{ if and only if } \tau(A) \in \mathsf{S}'.$$

Proof. The 'only if' direction follows from the fact that for all $A \in \mathsf{S}$, $\tau(A) \in \mathsf{S}_0$. Hence, as $\mathsf{S}' \supseteq \mathsf{S}_0$, $\tau(A) \in \mathsf{S}'$.

The 'if' direction follows from the fact that, for all $\tau(A) \in \mathsf{S}_1$, $A \in \mathsf{S}$. Hence if $\tau(A) \in \mathsf{S}'$, then as $\mathsf{S}_1 \supseteq \mathsf{S}'$ we know that $\tau(A) \in \mathsf{S}_1$ and thus that $A \in \mathsf{S}$. □

What the above results allow us to show is that $NRan(\tau, \mathsf{S})$ will have a particularly uniform structure. From Theorem 3.1.1 we know that – so long as $NRan(\tau, \mathsf{S}) \neq \emptyset$ – there will be a minimum logic in the set. By Zorn's Lemma we know that there will be some non-empty set of maximal logics within the range ($max(NRan(\tau, \mathsf{S}))$), and lastly by Theorem 3.3.6 that all the logics within $NRan(\tau, \mathsf{S})$ will be the extensions of the minimal logic which are sublogics of the logics in $max(NRan(\tau, \mathsf{S}))$. That is to say, the structure of the range of a translation τ over a source logic S is uniquely determined by the minimum logic, and the set $max(NRan(\tau, \mathsf{S}))$.

This general structure of the range of a translation leaves a number of degenerate cases. One of these degenerate cases we covered at the beginning of the chapter, namely that where $NRan(\tau, \mathsf{S}) = \emptyset$. What we will do now is briefly consider another one of the degenerate cases – where $NRan(\tau, \mathsf{S}) = \{\mathsf{S}\}$. Consider the translation $\tau_{\Box_D}$ which $\tau_{\Box_D}(\Box A) = \Box_D \tau_{\Box_D}(A)$ where $\Box_D A = \Box A \wedge \Diamond A$. Let **EN** be the smallest congruential modal logic containing the formula $\Box\top$.

Proposition 3.3.7. *If* $\mathsf{S}(\Box_D) \supseteq \mathbf{EN}$ *then* $\Diamond\top \in \mathsf{S}$.

Proof. $\mathsf{S}(\Box_D) \supseteq \mathbf{EN}$ implies that $\Box\top \in \mathsf{S}(\Box_D)$, which means that $\Box_D\top \in \mathsf{S}$ i.e. $\Box\top \wedge \Diamond\top \in \mathsf{S}$ – from which it follows that $\Diamond\top\mathsf{S}$. □

Theorem 3.3.8. **KD** *is maximal in* $NRan(\tau_{\Box_D}, \mathbf{KD})$.

Proof. Follows from the fact that $\Box_D A \leftrightarrow \Box A \in \mathbf{KD}$. $\Box$

Theorem 3.3.9. $NRan(\tau_{\Box_D}, \mathbf{KD}) = \{\mathbf{KD}\}$.

Proof. Suppose that $\mathsf{S} \in NRan(\tau_{\Box_D}, \mathbf{KD})$. Then as $\mathbf{KD} \supseteq \mathbf{EN}$ it follows from Proposition 3.3.7 that $\Diamond\top \in \mathsf{S}$ – i.e. that $\mathsf{S} \supseteq \mathbf{KD}$. By Theorem 3.3.8 we know that $\mathbf{KD}$ is maximal – and hence it follows that $NRan(\tau_{\Box_D}, \mathbf{KD}) = \{\mathbf{KD}\}$. $\Box$

3.3.1 A Strengthening of the Notion of the Range of a Translation

Let Tr be a translation, like T at the beginning of this chapter, which faithfully embeds IL into a modal logic S. In Chagrov & Zakharyashchev [1992, p.71] a modal logic S is said to be a *strong normal Tr-companion of* IL whenever, for all intuitionistic formulas A and sets of formulas Γ we have:[4]

$$A \in \mathrm{IL} + \Gamma \text{ if and only if } Tr(A) \in \mathsf{S} \oplus Tr(\Gamma). \tag{3.2}$$

Therein the question is raised as to what happens when, instead of merely considering the normal modal companions of IL, we instead consider what its strong normal modal companions are. It is well known, for example, that when we consider the the strong normal T-companions of IL that these are exactly the normal T-companions of IL, as shown in the following result.

Theorem 3.3.10 (Muravitsky [2006]). *For all normal modal logics M such that $\mathbf{S4} \oplus T(\Gamma) \subseteq M \subseteq \mathbf{Grz} \oplus T(\Gamma)$, and all intuitionistic formulas A we have the following.*

$$A \in IL + \Gamma \textit{ if and only if } T(A) \in M.$$

[4]We can recover the notion of what it is to for a modal logic S to be a *normal Tr-companion* of IL simply by making Γ the empty set. That is to say, a *normal Tr-companion* of IL is simply a normal modal logic into which Tr faithfully embeds IL.

We will have some further partial comments to make about the situation concerning modal companions in a moment, but mostly mention this as it raises an interesting question concerning a possible strengthening of the notion of the range of a translation in modal logic.

Let τ be a modal-to-modal translation, and S a normal modal logic, and define the set $SNRan(\tau, \mathsf{S})$ as the set of all normal modal logics S' such that, for all formulas A and B we have the following:

$$A \in \mathsf{S} \oplus B \text{ if and only if } \tau(A) \in \mathsf{S}' \oplus \tau(B). \tag{3.3}$$

The obvious question to ask at this point is in what ways will $SNRan(\tau, \mathsf{S})$ and $NRan(\tau, \mathsf{S})$ differ? It seems intuitive to think that the notion captured by $SNRan$ is in some sense stronger than that captured by $NRan$ – showing a kind of invariance under extensions of the source logic. On the intermediate logic front we know that the two notions converge when we consider the normal modal logics into which IL can be faithfully embedded by the translation T mentioned earlier. If we just consider the relationship between the S-companions and the strong S-companions of IL these two properties diverge, where S is the following translations from intuitionistic formulas to modal formulas.

$$\begin{array}{rcl} S(p_i) & = & \Box p_i \\ S(A \vee B) & = & S(A) \vee S(B) \\ S(A \wedge B) & = & S(A) \wedge S(B) \\ S(A \rightarrow B) & = & \Box(S(A) \rightarrow S(B)) \\ S(\neg A) & = & \Box \neg S(A). \end{array}$$

In particular consider the logics $\mathbf{S3} + X_i$, where $X_0 = \Diamond\bot$, $X_1 = \Box\Box\top$, $X_2 = \Box\Diamond\Diamond\bot$, and $X_{i+3} = \Diamond X_i \wedge \Diamond X_{i+1} \wedge \neg\Diamond X_{i+2}$. It is reported in [Chagrov & Zakharyashchev 1992, p.71] that all of the logics $\mathbf{S3} + X_i$ are modal S-companions of IL, but that none of the logics $\mathbf{S3}+X_i$ for $i \geq 3$ are strong S-companions of IL. Whether there is a normal modal logic S and translation

Tr such that S is a Tr-companion of IL, but not a strong Tr-companion of IL remains open.

We will now investigate what the structure of $SNRan$ is for a simple, and instructive case. Recall the following results.

Lemma 3.3.11 (Goris [2007]). $NRan(\tau_{\boxdot}, \mathbf{S5}) = [\mathbf{Kw4B}, \mathbf{S5}]$.

Proposition 3.3.12. *Suppose that* $\mathsf{S} \in NRan(\tau_{\boxdot}, \mathbf{S5})$. *Then for all formulas* A *and* B *we have the following.*

$$A \in \mathbf{S5} \oplus B \textit{ if and only if } \tau_{\boxdot}(A) \in \mathsf{S} \oplus \tau_{\boxdot}(B). \tag{3.4}$$

Proof. The 'if' direction follows by induction upon the length of derivations of A.

For the 'only if' direction suppose that $A \notin \mathbf{S5} \oplus B$, and thus that $\tau_{\boxdot}(A) \notin \mathbf{S5} \oplus B$. Then as $\mathbf{S5} \oplus B$ also proves $\tau_{\boxdot}(B)$, and thus $\mathsf{S} \oplus \tau_{\boxdot}(B) \subseteq \mathbf{S5} \oplus B$, it follows that $\tau_{\boxdot}(A) \notin \mathsf{S} \oplus \tau_{\boxdot}(B)$. □

Examining the above proof we can see that one particular fact about $\mathbf{S5}$ is used in the above proof – namely that $\mathbf{S5}$ is a maximal (indeed maximum) logic within $NRan(\tau_{\boxdot}, \mathbf{S5})$ with the following property, for all formulas A and B.

$$A \in \mathbf{S5} \oplus B \text{ if and only if } \tau_{\boxdot}(A) \in \mathbf{S5} \oplus \tau_{\boxdot}(B).$$

Given this, let us say that a logic $\mathsf{S}' \in max(NRan(\tau, \mathsf{S}))$ is *strongly maximal* whenever it fulfils (3.3). Then we are in a position to state our general result.

Proposition 3.3.13. *Suppose that* $\mathsf{S}' \in NRan(\tau, \mathsf{S})$, *and that all the maximal logics in* $NRan(\tau, \mathsf{S})$ *are strongly maximal. Then for all formulas* A *and* B *we have the following.*

$$A \in \mathsf{S} \oplus B \textit{ if and only if } \tau(A) \in \mathsf{S}' \oplus \tau(B).$$

Proof. The 'if' direction follows by induction upon the length of derivations of A.

For the 'only if' direction suppose that $A \notin \mathsf{S}$. Then, as S' is in $NRan(\tau, \mathsf{S})$ we know that there is some logic $\mathsf{S}'' \in max(NRan(\tau, \mathsf{S}))$ such that $\mathsf{S}' \subseteq \mathsf{S}''$. Then as S'' is strongly maximal we know that $\tau(A) \notin \mathsf{S}'' \oplus \tau(B)$, and consequently that $\tau(A) \notin \mathsf{S}' \oplus \tau(B)$. □

Corollary 3.3.14. *Suppose that all of the maximal logics in* $NRan(\tau, \mathsf{S})$ *are strongly maximal. Then* $SNRan(\tau, \mathsf{S}) = NRan(\tau, \mathsf{S})$.

The above corollary gives us a partial answer to the question concerning the difference between modal companions and strong modal companions in Chagrov & Zakharyashchev [1992]. If the maximal modal companions of IL relative to a translation Tr are strongly maximal (relative to that same translation), then nothing changes – the set of modal Tr-companions and the set of strong modal Tr-companions will coincide. We leave open the question as to what exactly can be said about the Range of a translation, intermediate or otherwise, whose maximal logics are not strongly maximal.

IV

The Range of Translations: Examples

What we will do in this chapter is go through and give some further examples of the range of translations for a selection of translations/source logic combinations which are of some independent interest – either from a technical or philosophical perspective.

4.1 The Range of Translation for KT!

Let us consider an easy example where we can give the full range of a modal translation τ. Recall that the logic **KT!** is the normal extension of **K** by the following axiom schema.

$$\mathbf{T!}: \quad \Box A \leftrightarrow A.$$

Rather than consider the range of a particular translation over **KT!** what we will do now is look at the range of all translations τ_∇ which replace $\Box$ with a linear modality $\nabla = O_1 \dots O_n$ with each $O_i \in \{\Diamond, \Box\}$. We will say that the *length* of such a modality ∇ is n. The first thing worth noting is that the smallest normal modal logic into which **KT!** can be faithfully embedded into is the following.

$$\mathbf{K}\nabla! : \quad \mathbf{K} \oplus \nabla p \leftrightarrow p.$$

Theorem 4.1.1. $\mathbf{K}\nabla!$ *is the smallest normal modal logic into which* **KT!** *can be faithfully embedded by* τ_∇.

Theorem 4.1.2. $\mathbf{K}\nabla! \nsubseteq \mathbf{KVer}$ *for all modalities* ∇ *of length* ≥ 1.

Proof. What we need to show is that $\mathbf{K}\nabla!$ is not valid on the singleton irreflexive frame $\mathfrak{F} = \langle \{w\}, \emptyset \rangle$. Firstly suppose that $O_1 = \Box$. Then we know that $\mathfrak{F} \models_w \nabla\bot$ and also that $\mathfrak{F} \not\models_w \bot$. Secondly suppose that $O_1 = \Diamond$. Then we know that $\mathfrak{F} \models_w \Box A$ while $\mathfrak{F} \not\models_w \nabla\Box A$, and the result follows. □

Theorem 4.1.3. **KT!** *is a maximal modal logic into which* **KT!** *can be faithfully embedded by* τ_∇.

Proof. Follows from the fact that $\tau_\nabla(A) \leftrightarrow A \in \mathbf{KT}!$. □

Theorem 4.1.4. $\mathsf{S} \in NRan(\tau_\nabla, \mathbf{KT}!)$ *if and only if* $\mathbf{K}\nabla! \subseteq \mathsf{S} \subseteq \mathbf{KT}!$.

Proof. For the 'only if' direction suppose that $\mathsf{S} \in R(\tau_\nabla, \mathbf{KT}!)$, and that $\mathsf{S} \nsubseteq \mathbf{KT}!$. Then by Theorem 2 of Makinson [1971] we know that $\mathsf{S} \subseteq \mathbf{KVer}$. But as $\mathsf{S} \in R(\tau_\nabla, \mathbf{KT}!)$ we know that $\mathsf{S} \supseteq \mathbf{K}\nabla!$ – from which it would follow that $\mathbf{K}\nabla! \subseteq \mathbf{KVer}$. But by Theorem 4.1.2 we know that this is not the case, and so $\mathsf{S} \subseteq \mathbf{KT}!$.

The 'if' direction follows trivially. □

For example, let us consider the structure of $R(\tau_{\Box\Box}, \mathbf{KT}!)$. In this case the minimum logic within the range will be the normal modal logic $\mathbf{KT}^2!$ which is the normal extension of $\mathbf{K}$ by the formula $\mathbf{T}^2!$.

$$\mathbf{T}^2! : \quad \Box\Box A \leftrightarrow A.$$

This logic can be easily seen to be determined by the class of frames $\langle W, R \rangle$ where $R^2(x) = \{x\}$ for all $x \in W$. It is in this guise that it is easy to see that this logic is none other than the normal modal logic $\mathbf{KB}_c$, as noted in Humberstone [2009].

$$\mathbf{B}_c: \quad A \to \Diamond\Box A.$$

This is very easy to see from a model-theoretic point of view. To see that this is the case syntactically first consider the following derivations which can be extracted from Williamson [1996].

(1)	$\Box\Box A \to \Diamond\Diamond A$	$\mathbf{T}^2!, \mathbf{K}$
(2)	$\Diamond\Diamond\neg A \vee \Diamond\Diamond A$	$(1), \mathbf{K}$
(3)	$\Diamond\top$	$(2), \mathbf{K}$
(4)	$\Box\Box A \to \Diamond\Box A$	$(3), \mathbf{K}$
(5)	$A \to \Box\Box A$	$\mathbf{T}^2!.$
(6)	$A \to \Diamond\Box A$	$(4), (5), TF.$

Now consider the following derivation of $\mathbf{T}^2!$ from $\mathbf{B_c}$.

(1)	$\Diamond\Box\top$	$\mathbf{B}_c, TF$
(2)	$\Diamond\top$	$(1), \mathbf{K}$
(3)	$\Box A \to \Diamond A$	$(2), \mathbf{K}$
(4)	$\Diamond\Box A \to \Diamond\Diamond A$	$(3), \Diamond - \mathbf{RM}$
(5)	$A \to \Diamond\Box A$	$\mathbf{B}_c$
(6)	$A \to \Diamond\Diamond A$	$(4), (5), TF$
(7)	$\Box A \to \Diamond\Box\Box A$	$\mathbf{B}_c$
(8)	$\neg\Box A \vee \neg\Box\neg\Box\Box A$	$(7), TF$
(9)	$\neg\Box(A \wedge \neg\Box\Box A)$	$(8), \mathbf{K}$
(10)	$\neg\Diamond\Box(A \wedge \neg\Box\Box A)$	$(9), \mathbf{RN}, \mathbf{K}$
(11)	$\Box\Diamond\neg(A \wedge \neg\Box\Box A)$	$(10), \mathbf{K}$
(12)	$\Box\Diamond\neg(A \wedge \neg\Box\Box A) \to \neg(A \wedge \neg\Box\Box A)$	$\mathbf{B}_c$

(13)	$\neg(A \wedge \neg\Box\Box A)$	(11), (12), TF
(14)	$\neg A \vee \Box\Box A$	(13), TF
(15)	$A \to \Box\Box A$	(14), TF
(16)	$A \leftrightarrow \Box\Box A$	(6), (15), TF

Thus we are able to conclude that $\mathbf{KT}^2! = \mathbf{KB}_c$, putting us in the position to conclude (with the help of Theorem 4.1.4) the following.

Corollary 4.1.5. $\mathsf{S} \in NRan(\tau_{\Box\Box}, \mathbf{KT}!)$ *if and only if* $\mathbf{KB}_c \subseteq \mathsf{S} \subseteq \mathbf{KT}!$.

As it happens we are in a position to strengthen this result somewhat. As $\mathbf{KB}_c$ is a proper extension of **KD!** we know, by corollary 2.3 of Segerberg [1986] that is is determined by a finite class of finite frames – in particular by the frames $\mathfrak{F}_1 = \langle\{x,y\},\{\langle x,y\rangle,\langle y,x\rangle\}\rangle$ and $\mathfrak{F}_2 = \langle\{x\},\{\langle x,x\rangle\}\rangle$. Furthermore, by corollary 2.3 we now that any consistent extension of $\mathbf{KB}_c$ must be characterized by a finite class of finite frames – which means it must either be determined by $\mathfrak{F}_1$ (which gives us $\mathbf{KB}_c$), or $\mathfrak{F}_2$, which gives us **T!**. Thus, as there are no logics properly between $\mathbf{KB_c}$ and $\mathbf{KT}!$ we can change the rhs of the above biconditional to "$\mathsf{S} \in \{\mathbf{KB}_c, \mathbf{KT}!\}$". Here we have an example (or more precisely a class of examples) where the set of normal modal logics into which a given source logic can be faithfully embedded by a given translation form an interval ordered under $\subseteq$ – the minimum such logic in this case being the one we have dubbed $\mathbf{K}\nabla!$, and the maximum such logic being **KT!** itself. That is to say, to use the nomenclature introduced earlier, the range of τ_∇ over **KT!** ($NRan(\tau_\nabla, \mathbf{KT}!)$) forms an interval.

4.2 The Range of $\tau_{\Diamond\Box}$ for KD45

This particular translation and source logic pair have some historical import, typically used with regards to epistemic and deontic logic. The first such result is that of Dawson [1959], which concerns the problem of trying to define a suitable deontic logic within the expressive resources of an alethic modal logic. What Dawson found was that, if you consider the modality '$\Diamond\Box$' in the modal logic **S4.2** as an abbreviation for a deontic 'O' operator, then the new operator fulfils all of the requirements given by A.R. Anderson for being a suitable deontic operator.[1] Later, in the rather short note Thomas [1967], Ivo Thomas shows that the $\{\Diamond\Box, \neg, \rightarrow\}$-fragment of **S4.2** is the logic **KD45**. This short note also shows that the translation which interprets $\Box A$ as $\Box A \wedge A$ (called $\tau_{\boxdot}$ in Zolin [2000]) faithfully embeds the modal logic **S4.4** into **KD45**.

Moving forward, we can see the same results cropping up again independently in Lenzen [1979], a paper concerned with epistemic logic. The idea there was that we could use the modality '$\Diamond\Box$' to define the belief operator within our epistemic logic – in essence defining the set of things an agent believes as those things which are compatible with what he knows. Lenzen showed that if we defined belief in this way in the epistemic logic **S4.4**[2] then the correct doxastic logic turned out to be **KD45**. This result, coupled with the result mentioned above about the translation $\tau_{\boxdot}$ faithfully embedding **KD45** into **S4.4** prompted Lenzen to believe that **S4.4** was the correct epistemic logic. These considerations were picked up again quite recently in Stalnaker [2006], where an attempt is made to characterize the various different concepts of knowledge given by the logics between **S4.2** and **S4.4**.

Another example of this translation in use is Byrd [1980], where an at-

[1] The requirements for a logic to be a suitable alethic modal logic are essentially that it is an extension of **KT** which does not prove $\mathbf{T}_c(p \rightarrow \Box p)$. The requirements for an operator 'O' to be a suitable deontic operator are essentially that the formula **D** is provable for 'O', and that **T** is not thus provable. That is to say, given a modal logic S and a modal function #, we can say that # is a suitable deontic operator if $\#p \rightarrow \neg\#\neg p \in \mathsf{S}$ and $\#p \rightarrow p \notin \mathsf{S}$.

[2] **S4.4** is also mentioned in the footnote Dawson [1959, p.78], where the lack of intuitive suitability of the deontic logic resulting from defining 'O' as '$\Diamond\Box$' in **S4.4** is attributed to Geach.

tempt is made to characterize the logic of the tense logical operator 'FG' – the future-directed tense logical version of $\Diamond\Box$. Following up on a conjecture of Rescher & Urquhart [1971], Byrd tries to show that the logic of 'eventual permanence' for linear time is the logic **KD5**. As mentioned in Humberstone [2006] though, this conjecture is incorrect in view of the **KD4.3**-provability of the formula '$\Diamond\Box p \to \Diamond\Box\Diamond\Box p$'. In French [2008] the present author has been able to show that the logic of 'eventual permanence' for linear time is **KD45**.

4.2.1 Preliminaries

Before we begin, we will find the following results and model constructions useful in what is to follow.

Given a transitive frame $\langle W, R\rangle$ say that $x \sim y$ for points $x, y \in W$ whenever $Rxy \vee Ryx \vee x = y$. Then a *cluster* is an equivalence class under $\sim$. We will say that a cluster is *degenerate* if it consists of an irreflexive singleton, *simple* if it contains only a reflexive singleton, and *non-degenerate* otherwise. Letting $C \geq_R D$ whenever $D \subseteq \bigcup_{x\in C} R(x)$, we will say that C *immediately succeeds* a cluster D whenever there is no C' such that $C \geq_R C' \geq_R D$. Finally a cluster is *first* if it is $\geq_R$-maximal, and *last* if it is $\geq_R$-minimal.

Proposition 4.2.1 (Segerberg [1971a, p.78]). **KD45** *is sound and complete with respect to the class of point generated frames $\mathfrak{F} = \langle W, R\rangle$ such that R is transitive, and $\mathfrak{F}$ contains either a single non-degenerate cluster, or a non-degenerate cluster which immediately succeeds a degenerate one.*

Proposition 4.2.2. *(Segerberg [1971a, p.156])* **S4.4** *is determined by the class of generated transitive frames with at most two clusters in which no cluster is degenerate and with at most the first cluster being simple.*

Proposition 4.2.3. *(Segerberg [1971a, p.77])* **S4.2** *is determined by the class of generated, transitive frames in which no cluster is degenerate and there is a last cluster.*

Definition 4.2.4. Given a set of points $C_n = \{x_1, \ldots, x_n\}$ define the frame $\mathfrak{F}_n = \langle W, R\rangle$ as follows.

- $W := C_n \cup \{w\}$.
- $R := C_n \times C_n \cup \{\langle w, x\rangle | x \in C_n\}$.

It is easy to see that the class of frames $\mathcal{C} = \{\mathfrak{F}_n | n \in Nat\} \cup \{\langle C_n, S_{C_n}\rangle | n \in Nat\}$ are all of the finite point generated frames for **KD45**, and that $\mathcal{C}^\circ$ – the reflexive closure of this class of frames – is the class of all finite point generated frames for **S4.4**.

Proposition 4.2.5. *For all formulas A, $\vdash_{\mathbf{KD45}} A \leftrightarrow \tau_{\Diamond\Box}(A)$.*

Proof. By induction upon the complexity of A, the only case of interest being in the induction step when $A = \Box B$ for some formula B. What we want to show then is that $\Box A \leftrightarrow \Diamond\Box A$. The right-to-left direction of this equivalence is just **5**. The left-to-right direction is derivable from **4** and **D** as follows: $\Box A \rightarrow \Box\Box A \rightarrow \Diamond\Box A$. By the inductive hypothesis this give us $\Box A \leftrightarrow \Diamond\Box\tau_{\Diamond\Box}(A)$ as desired. $\Box$

4.2.2 The Minimum Logic

We will begin our investigation of the structure of the range of $\tau_{\Diamond\Box}$ for **KD45** by first determining the minimal normal modal logic into which it can be embedded. It is easy to see that if a normal modal logic can be faithfully be embedded into a logic S by a translation τ_∇ then ∇ must be normal in S. Thus, we will begin our search for the minimal normal modal logic into which **KD45** can be faithfully embedded by $\tau_{\Diamond\Box}$ by looking at extensions of the smallest logic in which $\Diamond\Box$ is normal. In Humberstone [2006] it is shown that the smallest logic in which the modality $\Diamond\Box$ is normal is the logic **KDH**.

$$\mathbf{H}: \quad (\Diamond\Box p \wedge \Diamond\Box q) \rightarrow \Diamond\Box(p \wedge q).$$

Thus, it is easy to see that any logic into which **KD45** can be faithfully embedded must be an extension of **KDH**. This is a very soft lower bound, and as it happens we can strengthen this result somewhat further. What we will now show is that the minimum normal modal logic into which **KD45** can be faithfully embedded by $\tau_{\Diamond\Box}$ is the logic we will call $\mathbf{KDH4^{ml}5^{ml}}$, where $\mathbf{4^{ml}}$ and $\mathbf{5^{ml}}$ are the $\tau_{\Diamond\Box}$-translations of **4** and **5** given below.

$$\mathbf{4^{ml}}: \quad \Diamond\Box p \to \Diamond\Box\Diamond\Box p$$
$$\mathbf{5^{ml}}: \quad \Box\Diamond p \to \Diamond\Box\Box\Diamond p$$

Theorem 4.2.6. $\vdash_{\mathbf{KD45}} A$ *if and only if* $\vdash_{\mathbf{KD4.2}} \tau_{\Diamond\Box}(A)$.

Proof. The 'only if' direction follows from Proposition 4.2.5 and the fact that $\mathbf{KD4.2} \subseteq \mathbf{KD45}$, and the 'if' direction follows from the fact that $\mathbf{KD4.2} \subseteq \mathbf{KD45}$. □

Lemma 4.2.7. $\vdash_{\mathbf{KD4.2}} (\Diamond\Box p \wedge \Diamond\Box q) \to \Diamond\Box(p \wedge q)$.

Proof. We begin at line 4 of Dawson [1959, p.75], the first four lines being **K**-provable.

(1)	$\Diamond\Box p \to (\Diamond\neg\Box q \vee \Diamond\Box(p \wedge q))$	*Dawson*
(2)	$\Box\Box q \to (\Diamond\Box p \to \Diamond\Box(p \wedge q))$	(1), TF
(3)	$\Box q \to (\Diamond\Box p \to \Diamond\Box(p \wedge q))$	(2), **4**
(4)	$\Diamond\Box q \to \Diamond(\Diamond\Box p \to \Diamond\Box(p \wedge q))$	(3), **RM**$\Diamond$
(5)	$\Diamond\Box q \to (\Box\Diamond\Box p \to \Diamond\Diamond\Box(p \wedge q))$	(4), **K**
(6)	$\Box\Diamond\Box p \to (\Diamond\Box q \to \Diamond\Diamond\Box(p \wedge q))$	(5), TF
(7)	$\Diamond\Box\Box p \to \Box\Diamond\Box p$	**G**
(8)	$\Diamond\Box p \to \Diamond\Box\Box p$	**4**, $\Diamond$ – **EM**
(9)	$\Diamond\Box p \to \Box\Diamond\Box p$	(7), (8), TF
(10)	$\Diamond\Box p \to (\Diamond\Box q \to \Diamond\Diamond\Box(p \wedge q))$	(6), (10), TF
(11)	$(\Diamond\Box p \wedge \Diamond\Box q) \to \Diamond\Diamond\Box(p \wedge q))$	(10), TF
(12)	$(\Diamond\Box p \wedge \Diamond\Box q) \to \Diamond\Box(p \wedge q)$	(11), **4**.

$\Box$

Proposition 4.2.8. $\mathbf{KDH4^{ml}5^{ml}} \subseteq \mathbf{KD4.2}$

Proof. It is easy to see that $\mathbf{4^{ml}}$ and $\mathbf{5^{ml}}$ are both provable in **KD4.2**, as these are just $\tau_{\Diamond\Box}(4)$ and $\tau_{\Diamond\Box}(\mathbf{5})$. The only case left then is that of the **H** axiom which is covered by Lemma 4.2.7 $\Box$

Theorem 4.2.9. *For all formulas A:*

$$\vdash_{\mathbf{KD45}} A \textit{ if and only if } \vdash_{\mathbf{KDH4^{ml}5^{ml}}} \tau_{\Diamond\Box}(A) \tag{4.1}$$

Proof. The 'only if' direction proceeds by induction upon the length of derivations of A. For the base case suppose that A is an axiom. The case where A is **4** or **5** is handled by $\mathbf{4^{ml}}$ and $\mathbf{5^{ml}}$ respectively. The case where A is **K** or **D** follow from the fact that $\Diamond\Box$ is normal in **KDH** (and hence in $\mathbf{KDH4^{ml}5^{ml}}$) and that **KDH** proves **G** (which is the translation of **D**). The inductive step is trivial.

The 'if' direction, considered contrapositively, invites us to consider the case when $\nvdash_{\mathbf{KD45}} A$. By Theorem 4.2.6 this means that $\nvdash_{\mathbf{KD4.2}} \tau_{\Diamond\Box}(A)$. Thus, by Proposition 4.2.8 it follows that $\nvdash_{\mathbf{KDH4^{ml}5^{ml}}} \tau_{\Diamond\Box}(A)$. $\Box$

Theorem 4.2.10. $\mathbf{KDH4^{ml}5^{ml}}$ *is the minimum logic into which* **KD45** *can be faithfully embedded by $\tau_{\Diamond\Box}$.*

Proof. It is clear that every logic into which **KD45** can be faithfully embedded by $\tau_{\Diamond\Box}$ is an extension of **KDH** – as this is the smallest logic in which $\Diamond\Box$ is normal. Further, every such logic must prove $\tau_{\Diamond\Box}(4)$ and $\tau_{\Diamond\Box}(\mathbf{5})$ – which are axioms of $\mathbf{KDH4^{ml}5^{ml}}$. Hence, for all S, if $\mathsf{S}(\Diamond\Box) = \mathbf{KD45}$ then $\mathsf{S} \supseteq \mathbf{KDH4^{ml}5^{ml}}$. $\Box$

4.2.3 Maximal Logics

In this section we will investigate what the maximal logics into which **KD45** can be faithfully embedded by $\tau_{\Diamond\Box}$ are. In particular we will show that **KD45** and **S4.4** are amongst these logics. Before showing this though, we will first need to recall some results on the extensions of **KD45**. In particular, recall that the formula $\mathbf{Alt}_n$ for $n \in Nat$ is as follows, and is canonical for the condition on frames $\langle W, R\rangle$ that each point in W have no more than n R-successors (i.e. that $\forall x \in W : |R(x)| \leq n$).

$$\mathbf{Alt}_n : \quad \Box p_0 \vee \Box(p_0 \to p_1) \vee \ldots \vee \Box(p_0 \wedge \ldots \wedge p_{n-1} \to p_n).$$

Lemma 4.2.11. *Every modal logic* $\mathsf{S} \supseteq$ **KD45** *is either one of* **KD45**, **S5**, **Triv** *or one of* $\mathbf{KD45Alt}_n$, $\mathbf{S5Alt}_n$ *for some* $n \in \mathbb{N}$.

Proof. From Segerberg [1971a], p.127 we know that the above logics are all of the normal extensions of **KD45**. By the result listed on page 190 of Segerberg [1971a] we also know that every (quasi-normal) extension of **KD45** is normal, and hence will be equivalent to one of the logics listed above. □

Theorem 4.2.12 (**KD45** is maximal in $NRan(\tau_{\Diamond\Box}, \mathbf{KD45})$)**.** **KD45** *is faithfully embedded into a logic* $\mathsf{S} \supseteq$ **KD45** *by* $\tau_{\Diamond\Box}$, *then* $\mathsf{S} =$ **KD45**.

Proof. Suppose that a logic S is faithfully embedded into **KD45** by $\tau_{\Diamond\Box}$, and assume for a reductio that $\mathsf{S} \supsetneq$ **KD45**. By Lemma 4.2.11 we know that S is either an extension of **S5**, or $\mathbf{KD45Alt}_n$ for some $n \in Nat$. If $\mathsf{S} \supseteq$ **S5** then we know that $\Diamond\Box p \to p \in \mathsf{S}$, and that as this is just $\tau_{\Diamond\Box}(\Box p \to p)$ that the faithfulness of the translation would require that **KD45** prove **T**. If, on the other hand $\mathsf{S} = \mathbf{KD45Alt}_n$ for some $n \in Nat$ then as $\tau_{\Diamond\Box}(A) \leftrightarrow A \in \mathbf{KD45}$ for all A (and hence in S) we would have $\tau_{\Diamond\Box}(\mathbf{Alt}_n)$ provable in S, which by the faithfulness of the translation would require that **KD45** prove $\mathbf{Alt}_n$. As neither of these formulas are provable in **KD45** we are left with a contradiction, and the result follows. □

As one can see from examining the literature, most of the interest in the translation $\tau_{\Diamond\Box}$ has been in the interpretation of **KD45** as a doxastic logic, with the various target logics being envisaged as candidate epistemic logics in which we can define belief as possible knowledge.[3] To this end, we will turn now to looking for a maximal logic among the extensions of **KT**. A good place to start then is the logic **S4.4**, which is mentioned in Lenzen [1978] as being the strongest plausible epistemic logic. **S4.4** is the smallest normal extension of **S4** by the following formula.

$$.4: \quad p \to (\Diamond\Box p \to \Box p).$$

What we will now show is that this logic is maximal amongst the normal logics into which **KD45** is faithfully embedded by $\tau_{\Diamond\Box}$. Thus, if we are thinking of a plausible epistemic logic as being one in which we can define belief as possible knowledge, lending support to Lenzen's claim that **S4.4** is the strongest plausible epistemic logic. Before doing this we will first show that $\tau_{\Diamond\Box}$ faithfully embeds **KD45** into **S4.4**.

Proposition 4.2.13. *Let $\mathcal{M} = \langle W, R, V\rangle$ be a model on a serial, transitive frame, and let $\mathcal{M}^\circ = \langle W, R^\circ, V\rangle$ be the model resulting from extending R so that it is reflexive – i.e. $R^\circ = R \cup \{\langle x, x\rangle | x \in W\}$. Then, for all points $x \in W$ and all formulas A:*

$$\mathcal{M} \models_x \tau_{\Diamond\Box}(A) \iff \mathcal{M}^\circ \models_x \tau_{\Diamond\Box}(A).$$

Proof. By induction upon the complexity of A, the only case of interest being that where $A = \Box B$ for some formula B.

For the '$\Rightarrow$' direction suppose that $\mathcal{M} \models_x \Diamond\Box\tau_{\Diamond\Box}(B)$. Then, there exists a point $y \in R(x)$ such that $\mathcal{M} \models_y \Box\tau_{\Diamond\Box}(B)$. By the seriality of R we know that y has at least one R-successor, and that for all such R-successors z, $\mathcal{M} \models_z \tau_{\Diamond\Box}(B)$. Thus, by the induction hypothesis we can reason that, for all such $z \in W$, $\mathcal{M}^\circ \models_z \tau_{\Diamond\Box}(B)$. Additionally, we know that for all points $u \in R(y) \setminus \{y\}$ $\mathcal{M}^\circ \models_u \tau_{\Diamond\Box}(B)$. Moreover, by the transitivity of R (and hence R°) we know that every R-successor of a point $u \in R(y)$ such that $u \neq y$

[3] The notion of possibility involved here is epistemic possibility – i.e. the notion corresponding to $\Diamond$ for an epistemic reading of $\Box$. In particular we are thinking of 'possible knowledge' as the reading of the operator $\neg K \neg K$, where K is our epistemic $\Box$.

verifies $\tau_{\Diamond\Box}(B)$. Hence we can reason that, for all such $z \in R(y)$, $\mathcal{M}^\circ \models_z \Box\tau_{\Diamond\Box}(B)$. By transitivity we know that Rxz for all such z, and hence that $\mathcal{M}^\circ \models_x \Diamond\Box\tau_{\Diamond\Box}(B)$ as desired.

For the '$\Leftarrow$' direction suppose that $\mathcal{M}^\circ \models_x \Diamond\Box\tau_{\Diamond\Box}(B)$. Then there is a $y \in R^\circ(x)$ such that $\mathcal{M}^\circ \models_y \Box\tau_{\Diamond\Box}(B)$, and that for all $z \in R^\circ(y)$ we know that $\mathcal{M}^\circ \models_z \tau_{\Diamond\Box}(B)$. By the inductive hypothesis we know that for all such z, $\mathcal{M} \models_z \tau_{\Diamond\Box}(B)$. As $R^\circ(y) = R(y) \cup \{y\}$ we also know that all of the R-successors of y verify $\tau_{\Diamond\Box}(B)$, and thus that $\mathcal{M} \models_y \Box\tau_{\Diamond\Box}(B)$. Again, by the fact that $R^\circ(x) = R(x) \cup \{x\}$ we can see that Rxy, and consequently that $\mathcal{M} \models_x \Diamond\Box\tau_{\Diamond\Box}(B)$. □

It is worth noting that Proposition 4.2.13 allows us to show that **KD4** and **S4** are $\tau_{\Diamond\Box}$-equivalent. This result is not of present interest though in light of the fact **S4**($\Diamond\Box$) is not **KD45**, as **S4** fails to prove **G** (i.e. $\tau_{\Diamond\Box}(\mathbf{D})$).

Corollary 4.2.14. **KD45** $\equiv_{\tau_{\Diamond\Box}}$ **S4.4**.

Proof. The left-to-right direction follows from the fact that $\tau_{\Diamond\Box}(A) \leftrightarrow A \in$ **KD45**, and the fact that $\tau_{\Diamond\Box}$ faithfully embeds **KD45** into **S4.4**.

For the right-to-left direction suppose that $\tau_{\Diamond\Box}(A) \notin$ **KD45**. Then there is a model $\mathcal{M} = \langle W, R, V\rangle$ on a frame generated by a point w which either contains a single cluster, or a single irreflexive point which is R-related to a single cluster, and a point $x \in W$ such that $\mathcal{M} \not\models_x \tau_{\Diamond\Box}(A)$. By Proposition 4.2.13 this means that there is a model $\mathcal{M}^\circ \not\models_x \tau_{\Diamond\Box}(A)$. The only difference between $\mathcal{M}$ and $\mathcal{M}^\circ$ is that in $\mathcal{M}$ the generating point w is possibly irreflexive, while in $\mathcal{M}^\circ$ all points are reflexive. It is thus easy to see that $\mathcal{M}^\circ$ is a model for **S4.4** and thus that $\tau_{\Diamond\Box}(A) \notin$ **S4.4**. □

Proposition 4.2.15 (Lenzen [1979]). *For all formulas A:*

$$A \in \mathbf{KD45} \textit{ if and only if } \tau_{\Diamond\Box}(A) \in \mathbf{S4.4}.$$

Proof. Follows from Corollary 4.2.14 and the fact that $\tau_{\Diamond\Box}(A) \leftrightarrow A \in$ **KD45**. □

Lemma 4.2.16. *Every proper extension of* **S4.4** *proves one of (i)* **B**, *(ii)* $\mathbf{Alt}_n$, *or (iii)* $\mathbf{Alt}_n \vee \Diamond\Box p_{n+1} \to p_{n+1}$ *for some* $n \in Nat$.

Proof. This result follows from the inspection of the result in Segerberg [1971b] concerning the extensions of **S4.4** – (i) is provable in all extensions of **S5**, (ii) is provable in all the logics between **S4.4** and $\mathbf{S4Alt}_2$ (which is there called **V1**), (iii) is provable in all logics intermediate between **S4.4** and **S4.7**. □

Lemma 4.2.17. $\vdash_{\mathbf{KT}} \mathbf{Alt}_n \to \tau_{\Diamond\Box}(\mathbf{Alt}_n)$ *for all* $n \in Nat$.

Proof. $(\mathbf{Alt}_n) \to \Diamond(\mathbf{Alt}_n)$ by **T**, which distributing $\Diamond$ over the disjunction gives us $(\mathbf{Alt}_n) \to \tau_{\Diamond\Box}(\mathbf{Alt}_n)$. □

Theorem 4.2.18 (**S4.4** is maximal)**.** *If* $\tau_{\Diamond\Box}$ *faithfully embeds* **KD45** *into a logic* $\mathsf{S} \supseteq \mathbf{S4.4}$, *then* $\mathsf{S} = \mathbf{S4.4}$.

Proof. We proceed by showing that the translation $\tau_{\Diamond\Box}$ does not faithfully embed **KD45** into any proper extension of **S4.4**, the result then following from Proposition 4.2.15.

Suppose that some proper extension of **S4.4**, S, is such that $\tau_{\Diamond\Box}$ faithfully embeds **KD45** into S. By Lemma 4.2.16 we know that every proper extension of **S4.4** proves one of the following: (i) **B** , (ii) $\mathbf{Alt}_n$ for some $n \in Nat$, or (iii) $\mathbf{Alt}_n \vee (\Diamond\Box p_{n+1} \to p_{n+1})$. We proceed now by cases. For case (i) we are told that $\vdash_{\mathsf{S}} \Diamond\Box p \to p$. As this is the formula $\tau_{\Diamond\Box}(\Box p \to p)$, by the faithfulness of the translation we would have that $\vdash_{\mathbf{KD45}} \Box p \to p$. For case (ii) we are told that $\vdash_{\mathsf{S}} \mathbf{Alt}_n$ for some $n \in Nat$. By Lemma 4.2.17, and the fact that $\mathsf{S} \supseteq \mathbf{KT}$ we know that this means that $\vdash_{\mathsf{S}} \tau_{\Diamond\Box}(\mathbf{Alt}_n)$, which by the faithfulness of the translation would require that $\vdash_{\mathbf{KD45}} \mathbf{Alt}_n$, which is not the case. For the case of (iii) we note that this means that $\vdash_{\mathsf{S}} \mathbf{Alt}_n \vee (\Diamond\Box p_{n+1} \to p_{n+1})$. By Lemma 4.2.17 and classical reasoning this means that $\vdash_{\mathsf{S}} \tau_{\Diamond\Box}(\mathbf{Alt}_n) \vee (\Diamond\Box p_{n+1} \to p_{n+1})$, and that as this is $\tau_{\Diamond\Box}(\mathbf{Alt}_n \vee (\Box p_{n+1} \to p_{n+1}))$, the faithfulness of the translation would require that $\vdash_{\mathbf{KD45}} \mathbf{Alt}_n \vee (\Box p_{n+1} \to p_{n+1})$. As **KD45** proves none of these formulae, we can thus reason that **KD45** cannot be faithfully embedded into any proper extension of **S4.4**, and the result follows. $\square$

4.2.4 The Bigger Picture.

So far we have been probing the limits of the range of $\tau_{\Diamond\Box}$ for **KD45**, showing what its minimum is, and what its maximal logics are. What we will do now is to fill in the gaps in the structure of the range of $\tau_{\Diamond\Box}$ for **KD45**.

Definition 4.2.19. Given a model $\langle W, R, V\rangle$ generated by a point w with a last cluster $C \subseteq W$, construct the new model $\mathcal{M}_{(C,w)} = \langle W_{(C,w)}, R_{(C,w)}, V_{(C,w)}\rangle$ as follows:

- $W_{(C,w)} := C \cup \{w\}$.
- $R_C := (R \cap C \times C) \cup \{\langle w, x\rangle | x \in W_{(C,w)}\}$.
- $V_{(C,w)}(p_i) := V(p_i) \cap W_{(C,w)}$.

Definition 4.2.20. $\mathcal{M}, C \models A \iff \forall x \in C, \mathcal{M} \models_x A$.

Lemma 4.2.21. *Let $\mathcal{M} = \langle W, R, V\rangle$ be a model on a transitive frame with a last cluster $C \subseteq W$. Then for all formulas A, and all $x \in W$:*

$$\mathcal{M} \models_x \Diamond\Box A \iff \mathcal{M}, C \models A. \tag{4.2}$$

Proof. ($\Rightarrow$) Suppose $\mathcal{M} \models_x \Diamond\Box A$. Then there is a point $y \in W$ such that Rxy and $\mathcal{M} \models_y \Box A$. This means that for all points $z \in R(y)$, $\mathcal{M} \models_z A$. Thus, by the definition of a last cluster we know that $C \subseteq R(y)$, and hence $\mathcal{M}, C \models A$ as desired.

($\Leftarrow$) Suppose that $\mathcal{M}, C \models A$. As C is last in $\mathcal{M}$ we know that for all $y \in C$, $R(y) = C$. Hence $\mathcal{M}, C \models \Box A$, and thus by the definition of a last cluster we know that $\mathcal{M} \models_x \Diamond\Box A$. $\square$

Theorem 4.2.22. *Suppose that $\mathcal{M} = \langle W, R, V\rangle$ is a model generated by a point w on a reflexive, transitive frame with a last cluster C, and that $\mathcal{M}_{(C,w)}$ =*

$\langle W_{(C,w)}, R_{(C,w)}, V_{(C,w)}\rangle$ is a model obtained via Definition 4.2.19. Then for all formulas A:

$$\mathcal{M} \models_w \tau_{\diamond\Box}(A) \iff \mathcal{M}_{(C,w)} \models_w \tau_{\diamond\Box}(A). \tag{4.3}$$

Proof. By induction upon the complexity of A, the only case of interest being that where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that that $\mathcal{M} \models_w \Diamond\Box\tau_{\diamond\Box}(B)$. By Lemma 4.2.21 it follows that $\mathcal{M}, C \models \tau_{\diamond\Box}(B)$. By the induction hypothesis this means that, for all such points $y \in C$, $\mathcal{M}_{(C,w)} \models_y \tau_{\diamond\Box}(B)$. As for all of these points $R_{(C,w)}(y) = C$ we can see that $\mathcal{M}_{(C,w)} \models_y \Box\tau_{\diamond\Box}(B)$, and that by the definition of $R_{(C,w)}$ that $R_{(C,w)}wy$, and thus $\mathcal{M}_{(C,w)} \models_w \Diamond\Box\tau_{\diamond\Box}(B)$.

For the 'if' direction suppose that $\mathcal{M}_{(C,w)} \models_w \Diamond\Box\tau_{\diamond\Box}(B)$. This means that $\mathcal{M}_{(C,w)}, C \models \tau_{\diamond\Box}(B)$. By the induction hypothesis this means that for all points $y \in C$, $\mathcal{M} \models_y \tau_{\diamond\Box}(B)$, which by Lemma 4.2.21 means that $\mathcal{M} \models_w \Diamond\Box\tau_{\diamond\Box}(B)$ as desired. □

Theorem 4.2.23. **S4.2** $\equiv_{\tau_{\diamond\Box}}$ **S4.4**.

Proof. The right-to-left direction follows from the fact that **S4.2** $\subseteq$ **S4.4**. To show that the reverse direction holds, suppose that $\tau_{\diamond\Box}(A) \notin$ **S4.2** for some formula A. Then by Proposition 4.2.3 there is a model $\mathcal{M}$ on a transitive frame generated by a point w with a last cluster, and $\mathcal{M} \not\models_w \tau_{\diamond\Box}(A)$. Then by Theorem 4.2.22 there is a model $\mathcal{M}_{(C,w)}$ such that $\mathcal{M}_{(C,w)} \not\models_w \tau_{\diamond\Box}(A)$. It is easy to see that $\mathcal{M}_{(C,w)}$ has only two clusters, with the first being simple, and that thus by Proposition 4.2.2 that $\tau_{\diamond\Box}(A) \notin$ **S4.4** as desired. □

Theorem 4.2.24. *For all logics* S*, such that* **S4.2** $\subseteq \mathsf{S} \subseteq$ **S4.4***:*

$$\mathbf{S4.2} \equiv_{\tau_{\diamond\Box}} \mathsf{S} \equiv_{\tau_{\diamond\Box}} \mathbf{S4.4}. \tag{4.4}$$

Corollary 4.2.25. **KD4.2** $\equiv_{\tau_{\diamond\Box}}$ **S4.2**.

Proof. The left-to-right direction follows from the fact that **KD4.2** $\subseteq$ **S4.2**. For the right-to-left direction, suppose that $\tau_{\diamond\Box}(A) \notin$ **KD4.2**. Then there is a model $\mathcal{M} = \langle W, R, V\rangle$ on a serial, transitive and convergent frame, and a point $x \in W$ such that $\mathcal{M} \not\models_x \tau_{\Diamond\Box}(A)$. By Proposition 4.2.13 we know that $\mathcal{M}^\circ \not\models_x \tau_{\Diamond\Box}(A)$. As this model is just the reflexive version of $\mathcal{M}$ this model is one on a reflexive, transitive and convergent frame, and thus as **S4.2** is complete w.r.t this class of frames we can conclude that $\tau_{\Diamond\Box}(A) \notin \mathbf{S4.2}$. $\Box$

Theorem 4.2.26. *For all logics* S*, such that* $\mathbf{KD4.2} \subseteq \mathsf{S} \subseteq \mathbf{KD45}$*:*

$$\mathbf{KD4.2} \equiv_{\tau_{\Diamond\Box}} \mathsf{S} \equiv_{\tau_{\Diamond\Box}} \mathbf{KD45}. \tag{4.5}$$

4.2.5 Maximal Logics in $NRan(\tau_{\Diamond\Box}, \mathbf{KD45})$ – a partial solution

What we will do here is give a partial solution to the problem of determining what the logics in $max(NRan(\tau_{\Diamond\Box}, \mathbf{KD45}))$ are. Our solution is partial in two different respects. Firstly we are only considering those logics which are Kripke complete – that is to say, determined by some class of frames. Secondly we are only considering those logics which are extensions of the modal logic **KD4.2**.

Recall the frames $\mathfrak{F}_n$ from Definition 4.2.4.

Proposition 4.2.27. *Suppose that* $\mathfrak{F}_n$ *and* $\mathfrak{F}_m$ *are such that* $m > n$*. Then there is a p-morphism* f *from* $\mathfrak{F}_m$ *onto* $\mathfrak{F}_n$*.*

Proof. Let f be any onto function which maps w onto w, and points in C_m onto points in C_n such that $f^{-1}(C_n) = C_m$. $\Box$

Proposition 4.2.28. *Suppose that* $\mathfrak{F}_n^\circ$ *and* $\mathfrak{F}_m^\circ$ *are such that* $m > n$*. Then there is a p-morphism* f *from* $\mathfrak{F}_m^\circ$ *onto* $\mathfrak{F}_n^\circ$*.*

Proof. As for Proposition 4.2.27. $\Box$

We will also require the following formula $\mathbf{X}_n$, which is valid at a point x in a frame $\langle W, R\rangle$ iff $|R(x)| \geq n$.

$$\mathbf{X}_n : \quad \bigwedge_{0 \leq i < n} \Diamond(p_0 \wedge \ldots \wedge \neg p_i \wedge \ldots \wedge p_{n-1}).$$

Theorem 4.2.29. *Suppose that* S *is a Kripke complete normal modal logic extending* **KD4.2** *such that* $\mathsf{S} \nsubseteq$ **KD45** *and* $\mathsf{S} \nsubseteq$ **S4.4**. *Then, for some* $n \in Nat$ *we have the following.*

$$\Diamond\Box\mathbf{X}_n \to (\Diamond\Box p_{n+1} \to p_{n+1}) \in \mathsf{S}.$$

Proof. As $\mathsf{S} \nsubseteq$ **KD45** we know that there if a frame $\mathfrak{F}_n$ such that $\mathfrak{F}_n \notin Fr(\mathsf{S})$, and likewise as $\mathsf{S} \nsubseteq$ **S4.4** we know that there is a frame $\mathfrak{F}^\circ_m \notin Fr(\mathsf{S})$. Letting $k = max(m,n)+1$, suppose for a contradiction that $\Diamond\Box\mathbf{X}_k \to (\Diamond\Box p_{k+1} \to p_{k+1}) \notin \mathsf{S}$. As S is Kripke complete this would mean that there is a frame $\mathfrak{G} = \langle U,S\rangle$ in $Fr(\mathsf{S})$ and a valuation V such that, for some point $x \in U$ we have $\langle U,S,V\rangle \models_x \Diamond\Box X_k$ and $\langle U,S,V\rangle \not\models_x \Diamond\Box p_{k+1} \to p_{k+1}$. As $\langle U,S,V\rangle \models_x \Diamond\Box X_k$ it follows by Lemma 4.2.21 that $\langle U,S,V\rangle, C \models X_k$. So the final cluster, C, of $\mathfrak{G}$ contains at least $n+m+1$ points. Suppose that there is a point $x \in U \setminus C$ such that $R(x) = C$. Then as $Fr(\mathsf{S})$ is closed under point generated subframes this would mean that $\mathfrak{F}_{n+m+1} \in Fr(\mathsf{S})$. As $n+m+1 > n$ it follows by Proposition 4.2.27 that $\mathfrak{F}_n \in Fr(\mathsf{S})$ – contrary to our hypothesis. So there is no point $x \in U \setminus C$ such that $R(x) = C$. Consider now the p-morphism $f : U \to W$ from $\langle U,S\rangle$ onto $\mathfrak{F}^\circ_k$. which maps each point in C to a point in C_k, and all other points onto w. It is clear that this is a p-morphism from $\mathfrak{G}$ onto $\mathfrak{F}^\circ_k$. As $n+m+1 > m$ it the follows by Proposition 4.2.28 that $\mathfrak{F}^\circ_m \in Fr(\mathsf{S})$ – contrary to our hypothesis. Consequently there can be no such frame $\mathfrak{G}$ and the result follows. □

Theorem 4.2.30. *The set of all Kripke complete maximal logics extending* **KD4.2** *in* $NRan(\tau_{\Diamond\Box},$ **KD45**$)$ *consists of exactly the logics* **KD45** *and* **S4.4**

Proof. Follows from Theorem 4.2.29 and the fact that $\Diamond\Box\Box X_n \leftrightarrow \tau_{\Diamond\Box}(\Box X_n)$ is provable in **K** and that $\Diamond\Box A \leftrightarrow \Diamond\Box\Box A$ is provable in **KD4.2**.

□

What this means then is that if there are any maximal normal modal logics in $NRan(\tau_{\Diamond\Box},$ **KD45**$)$ other than **KD45** and **S4.4** then either they are not extensions of **KD4.2**, or they are not Kripke complete.

4.3 The KT-embedding Problem

The most common translation occurring in the literature is the translation $\tau_{\boxdot}$, which replaces all occurrences of $\Box A$ in a formula with $\boxdot A$ (alias. $\Box A \wedge A$). There are obvious philosophical reasons for being interested in this translation – arising, for example, from considerations of doxastic and provability logics – the second of which we will mention here. In provability logic we are interested in modal logics in which we can interpret $\Box A$ as meaning that, for every function * which maps the propositional variables of A to formulas of Peano Arithmetic (PA for short), the sentence A^* is provable in PA. As it turns out this logic – the logic of provability in PA – is the normal modal logic **GL**. One fact which we know about provability in PA is that there are some sentences which are true, but which are not provable – this following from Gödel's first incompleteness theorem. The question then arises of finding the modal logic S for which $\Box A$ can be interpreted as meaning that A^* is both true and provable in PA – i.e. of finding the logic which is embedded into **GL** by the translation $\tau_{\boxdot}$. As it turns out this logic, the logic of *strong provability* as this modal operator is often referred to in provability logic, is the normal modal logic **Grz**. Equally though, we might want to know what the logic of strong provability is when we interpret $\Box A$ as being provability in some weaker (or even perhaps stronger) arithmetic theory than PA – and so it is worthwhile investigating the inferential behaviour of $\boxdot$, and by extension the behaviour and properties of the $\tau_{\boxdot}$-translation in a general setting, as this will by extension allow us to discover some things about strong provability in arithmetic theories other than PA.

This translation also connects up with the Kripke semantics in a very intuitive way – a formula of the form $\tau_{\boxdot}(A)$ being true at a point x in a model $\mathcal{M} = \langle W, R, V \rangle$ iff the formula A is true at x in the model $\mathcal{M}^\circ = \langle W, R^\circ, V \rangle$ – where R° is the reflexive closure of the relation R. Similarly, a formula $\tau_{\boxdot}(A)$ is valid on a frame $\langle W, R \rangle$ just in case A is valid on the frame $\langle W, R^\circ \rangle$.

One of the interesting features of this translation stems from the following result.

Theorem 4.3.1. *For all formulas A we have the following.*

$$A \in \mathbf{KT} \text{ if and only if } \tau_{\boxdot}(A) \in \mathbf{K}.$$

Now there is a sense in which the $\tau_{\boxdot}$-translation is an obvious one, having been 'read off' the **T** axiom. One might reason that what the **T** axiom tells us is that when a formula $\Box A$ is true at a point in a model, then A is also true at that point in the model. So we reason that we can mimic the behaviour of the **KT**-box operator in **K** using $\boxdot$. One might then wonder whether we can extend this reasoning to any normal extension of **K** by an axiom of the following form, for some formula in a single variable $X(p)$.

$$\Box p \to X(p).$$

The idea here would be to use a modal translation τ such that $\tau(\Box A) = \Box\tau(A) \wedge X(\tau(A))$. If this approach worked then we would have a recipe for translating many common modal logics into **K** – for example **4** is an axiom of the above form where $X(p) = \Box\Box p$, and **D** is one where $X(p) = \Diamond p$. Unfortunately, this approach (to translating normal modal logics faithfully into **K**) does not work in general –one of the main purposes of the translation $\tau_{\Box_D}$ in section 3.3 being to show this. What the $\tau_{\Box_D}$ translation tells us is that, not only does $\tau_{\Box_D}$ not faithfully embed **KD** into **K**, but it doesn't embed **KD** into any logic distinct from **KD** itself! Similarly, we can show that this procedure does not work where $X(p) = \Box\Box p$ – the translation of the **4**-axiom failing to be **K**-provable. This idea is recaptured somewhat in the non-compositional translation of **K4** into **K** given in Fitting [1988].

One obvious thing to wonder then is what the range of $\tau_{\boxdot}$ is for **KT**. This is an obvious thing to wonder mostly due to the prevalence and simplicity of the translation involved, its obvious connection to the Kripke semantics, and the simplicity of the source logic involved. As it turns out this is no simple task, and in this section we will outline some known results as to the structure of $NRan(\tau_{\boxdot}, \mathbf{KT})$ as well as providing some weak evidence as to what its structure might be.

We will begin by restating some well known results.

Theorem 4.3.2. **KT** *is maximal in* $NRan(\tau_{\boxdot}, \mathbf{KT})$.

Proof. Follows by theorem 3.2.1 and the fact that $\tau_{\boxdot}(A) \leftrightarrow A \in \mathbf{KT}$. □

Theorem 4.3.3. **K** *is minimal in* $NRan(\tau_{\boxdot}, \mathbf{KT})$.

Proof. Follows from the fact that **K** is the smallest normal modal logic, and thus – as it is in $NRan(\tau_{\boxdot}, \mathbf{KT})$ – it must be the smallest such logic. □

So it follows by our convexity principle (Theorem 3.3.6) that all the normal modal logics between **K** and **KT** are in $NRan(\tau_{\boxdot}, \mathbf{KT})$. What we will do in the rest of this section is provide some evidence for the following conjecture.

Conjecture 4.3.4 (**KT** Embedding Problem). $NRan(\tau_{\boxdot}, \mathbf{KT})$ *consists of all and only the modal logics in the interval* $(\mathbf{K}, \mathbf{KT})$.

As **K** is the minimal logic into which **KT** can be faithfully embedded by $\tau_{\boxdot}$, we are able to clarify what is needed in order to prove the correctness of the above Conjecture. What we need to show is that for every formula A such that $A \notin \mathbf{KT}$ that there is a formula B such that $\tau_{\boxdot}(B) \in \mathbf{K} \oplus A$. For some logics it is very easy to find the desired formula B – $\tau_{\boxdot}(A)$ being provable in $\mathbf{K} \oplus A$. Good examples of such formula A for which we have this behaviour are **4** and **B**. Moreover, for these logics the following result allows us to determine what the logic $\mathsf{S}(\boxdot)$ is.

Theorem 4.3.5 (Zolin [2000, p.881]). *If* S *is a normal modal logic such that* $\mathsf{S}(\boxdot) \supseteq \mathsf{S}$ *then* $\mathsf{S}(\boxdot) = \mathsf{S} \oplus \{\Box p \to p\}$.

For the majority of formulas A, though, we are going to have to find a formula B which is distinct from A whose $\boxdot$-translation is provable in $\mathbf{K} \oplus$

A. One place which provides some suggestive information regarding such formulas B is the Kripke semantics. One thing we will often find when a formula A is not **KT**-provable then the reflexive closure of the frames on which A is valid prove some formula B which is not **KT**-provable. For example consider the case where A is the formula $\mathbf{D}_c$. $\mathbf{D}_c$ is valid on the class of all frames which are *partially functional* in the sense that each point x has at most one R-successor. The reflexive closure of the class of all such frames satisfies the property that each point has at most two R-successors – making the formula $\mathbf{Alt}_2$ valid. Consequently we know that the formula $\tau_{\boxdot}(\mathbf{Alt}_2) \in \mathbf{KD}_c$ and thus that this logic is not in $NRan(\tau_{\boxdot}, \mathbf{KT})$. Using this method (of inspection of Kripke frames) we can show that all the common normal modal logics are not in $NRan(\tau_{\boxdot}, \mathbf{KT})$ – as shown in Table 4.3.

S'	**KT**-unprovable formula A s.t. $\tau_{\boxdot}(A) \in \mathsf{S}'$.
K5	$(p \wedge \neg q) \to (\neg\Diamond\Box(p \wedge \neg q) \vee \neg\Diamond\Box(\neg p \wedge \neg q))$.
K4	$\Box p \to \Box\Box p$.
KB	$p \to \Box\Diamond p$.
$\mathbf{KT}_c$	$p \to \Box p$.
$\mathbf{K}\oplus\{\Diamond\Box p \to \Box\Diamond p\}$	$\Diamond\Box p \to \Box\Diamond p$.
$\mathbf{K}\oplus\{\Box\bot \vee \Diamond\Box\bot\}$	$\Box\Diamond p \to \Diamond\Box p$.
$\mathbf{KAlt}_n$	$\mathbf{Alt}_{n+1}$.

Table 4.1: A list of common modal logics, and a formula A which is not a theorem of **KT** for which they prove $\tau_{\boxdot}(A)$.

4.3.1 Formulas of modal degree one

What we will do here is present a partial confirmation of the conjecture given above, given in French & Humberstone [2009]. That is, we will show that when A is a formula of modal degree 1 which is not **KT**-provable, that the logic $\mathbf{K} \oplus A$ proves a formula $\tau_{\boxdot}(B)$ such that $B \notin \mathbf{KT}$. Recall that every modal formula A is equivalent to a conjunction of formulas $A_1 \wedge \ldots \wedge A_k$

where each of the A_i is of the following form.

$$(B \wedge \Box C) \to (\Box D_1 \vee \ldots \vee D_n).$$

In the case where A is of modal degree one we know that each of the formulas $B, C, D_1, \ldots, D_n$ will all be of modal degree zero – containing no $\Box$-operators, and as such are all equivalent to their $\tau_{\boxdot}$-translations. Let us define a function f such that is A is a set of 'basic disjunctions' $A_1 \wedge \ldots \wedge A_k$ as above, then $f(A) = f(A_1) \wedge \ldots \wedge f(A_k)$ where $f(A_i)$ is the formula:

$$(s \wedge B \wedge \boxdot C) \to (\boxdot(D_1 \vee s) \vee \ldots \vee \boxdot(D_n \vee s)).$$

Here we are taking s as the first propositional variable not occurring in A – although all the matters here is that s be a new propositional variable.

Lemma 4.3.6. *For all formulas A we have that $A \to f(A) \in$* **K**.

Proof. Follows from that fact that, where $\bigwedge_{i=1}^{k} A_i$ is a normal form for A, $A_i \to f(A_i) \in$ **K**. $\Box$

Lemma 4.3.7. *For all formulas A of modal degree 1 we have that if $f(A) \in$* **K** *then $A \in$* **KT**.

Theorem 4.3.8. *If $\mathsf{S} = \mathbf{K} \oplus \Gamma$ for Γ a set of formulas of modal degree 1 and $\mathsf{S} \nsubseteq \mathbf{KT}$ then $\tau_{\boxdot}$ does not faithfully embed* **KT** *into* S.

This allows us to settle the **KT**-embedding Conjecture as it bears upon normal extensions of **K** by formulas of modal degree at most 1 in the affirmative. All such logics which are not sublogics of **KT** proving some **KT**-unprovable $\boxdot$-formula. This of course does not settle the conjecture one way or the other, and what we will close this chapter with is a brief suggestion of one way in which this might be done.

Let us briefly consider the **5** axiom ($= \Diamond p \to \Box \Diamond p$). Put into conjunctive normal form this is equivalent to the following formula $A_{\mathbf{5}}$.

$$A_{\mathbf{5}}: \quad \Box \neg p \vee \Box(\Box \neg p \to \bot).$$

This is just a basic disjunction where there are no formulas B or C and $n = 2$ – the formula under the scope of $\Box$ in the second disjunct being of the form $\Box C \rightarrow \bot$. As this is a formula of modal degree 2 we cannot apply the function f above to our normal form and a get a desirable result. Instead consider the following class of functions f_k, where $f_k(A_i)$ is as follows.

$$(p_k \wedge f_{k+1}(B) \wedge \boxdot f_{k+1}(C)) \rightarrow (\boxdot(f_{k+1}(D_1) \vee p_k) \vee \ldots \vee \boxdot(f_{k+1}(D_n) \vee p_k)).$$

If we assume that our basic disjunctions A_i are all constructed out of some set of propositional variables $\{p_0, \ldots, p_m\}$, then the suggestion being made here is that we apply the above function f_k where $k = m+1$. Whether this suggestion could be shown to work in the general case is not clear, but what we will offer here is some (merely) suggestive evidence.Consider the formula $f_1(A_5)$.

$$f_1(A_5): \quad p_1 \rightarrow (\boxdot(\neg p_0 \vee p_1) \vee \boxdot((\boxdot\neg p_0 \wedge p_2 \rightarrow \bot) \vee p_1).$$

Now it can easily be shown that $f_1(A_5) \in \mathbf{K5}$ and that $f_1(A_5) = \tau_{\boxdot}(B)$ for a formula $B \notin \mathbf{KT}$. So one way we might consider proceeding is along the above lines – the present author has been unable to get any general results along these lines, though.

V

Intertranslatability and Translational Equivalence

In this chapter we will look at a cluster of relations between logics S and S' which are stronger than that of S being able to be faithfully embedded into S'. Predominantly we will be concerned with arbitrary modal logics and the modal-to-modal translations between them, although it should be clear to the reader how to transfer many of the results here to the case where arbitrary logics (considered as sets of formulas) are at issue. The obvious place for us to start our investigation is to consider the relation of intertranslatability. Let us say that two logics S and S' are *intertranslatable via τ and τ'* whenever τ faithfully embeds S into S' and τ' faithfully embeds S' into S. We will say that S and S' are intertranslatable simpliciter whenever there exist translations τ and τ' such that S and S' are intertranslatable via τ and τ' respectively. Throughout this chapter we will adopt the convention of associating translations with source logics – hence τ is a translation which faithfully embeds S into S' and likewise for τ' and

S'. The notion of two logics being intertranslatable is rarely considered in isolation – usually being mentioned only as an afterthought to discussions of translational equivalence (which we will get to shortly) – but it is instructive to consider how far we can get by just assuming intertranslatability and not translational equivalence (see footnote 4 for example). Firstly, it is quite easy to show that intertranslatable logics constitute the equivalence classes of the relation 'is faithfully embeddable into'. That is to say, suppose that we define the following relation between logics S and S': S *is faithfully embeddable into* S', or equally that S *can be faithfully embedded into* S', whenever there exist a translation τ which faithfully embeds S into S'. Then the sets of logics which are equivalent under this relation are the intertranslatable ones. That is to say, logics which are intertranslatable agree upon where they stand in the 'is faithfully embeddable into' relation, a fact which we will now prove.

Theorem 5.0.9. *Suppose that* S *and* S' *are intertranslatable via* τ *and* τ'. *Then for all logics* S'', S'' *can be faithfully embedded into* S *if and only if it can be faithfully embedded into* S'.

Proof. For the 'only if' direction suppose that t faithfully embeds S'' into S, and that $A \in \mathsf{S}''$. Then we know that $t(A) \in \mathsf{S}$, and thus by the fact that τ faithfully embeds S into S' we have that $\tau(t(A)) \in \mathsf{S}'$. Consequently we can see that S'' can be faithfully embedded into S' by the translation $\tau \circ t$.

For the 'if' direction suppose that t' faithfully embeds S'' into S', and that $A \in \mathsf{S}''$. Then by supposition we know that $t'(A) \in \mathsf{S}'$, and thus by the fact that τ' faithfully embeds S into S' we have that $\tau'(t'(A)) \in \mathsf{S}'$. Consequently we can see that S'' can be faithfully embedded into S' by the translations $\tau' \circ t'$. $\square$

Theorem 5.0.10. *Suppose that* S *and* S' *are intertranslatable via* τ *and* τ'. *Then* S *can be faithfully embedded into a logic* S'' *if and only if* S' *can be faithfully embedded into* S''.

Proof. For the 'only if' direction suppose that t faithfully embeds S into S'' and that $A \in \mathsf{S}'$. Then we know that $\tau'(A) \in \mathsf{S}$, and thus that $t(\tau'(A)) \in \mathsf{S}''$. Thus we can see that S' can be faithfully embedded into S'' by the translation $t \circ \tau'$.

For the 'if' direction suppose that t' faithfully embeds S' into S'', and that $A \in \mathsf{S}$. Then we know that $\tau(A) \in \mathsf{S}'$, and thus that $t'(\tau(A)) \in \mathsf{S}''$. Thus we can see that S can be faithfully embedded into S'' by the translation $t' \circ \tau$. □

What this tells us so far is that logics which are intertranslatable are equivalent with respect to translations, and that the relation of 'is intertranslatable with' is an equivalence relation on the set of all logics. As mentioned above though, this relation has not historically been the one with which people have been primarily interested. Let us consider now a particular strengthening of the notion of intertranslatability. Let us say that S and S' are *translationally equivalent via τ and τ'* whenever S and S' are intertranslatable via τ and τ' and additionally fulfil the following condition for all formulas A.

$$\tau'(\tau\ (A)) \leftrightarrow A \in \mathsf{S} \tag{5.1}$$

$$\tau\ (\tau'(A)) \leftrightarrow A \in \mathsf{S}' \tag{5.2}$$

Here we are taking '$\leftrightarrow$' to be a connective common to the languages of both S and S' which acts as an equivalence connective in a way which we will now specify. Say that two formulas A and B are *synonymous* according to a logic S whenever, for all contexts $C(\cdot)$, if $C(A) \in \mathsf{S}$ then $C(B) \in \mathsf{S}$ where $C(B)$ results from $C(A)$ by replacement of one or more occurrences of A with B.[1] Following the usage given in Humberstone [2005b] let us

[1]This is synonymy in the sense of Smiley [1962], and is one of many reasons why the present author thinks that it is ill-advised to call what we are calling translational equivalence 'synonymy', as is done in Pelletier & Urquhart [2003] and de Bouvère [1965].

say that a logic S is *equivalential* if there is a set of formulas (called *equivalence formulas*) $\Delta(p,q)$ constructed solely out of the propositional variables p and q such that A and B are synonymous whenever $\Delta(A,B) \subseteq \mathsf{S}$.[2] Two logics S and S' are *similarly equivalential* if there is a set of formulas $\Delta(p,q)$ such that A and B are synonymous according to S (resp. S') iff $\Delta(A,B) \subseteq \mathsf{S}$ (resp. $\Delta(A,B) \subseteq \mathsf{S}'$). In saying that the connective '$\leftrightarrow$' used in the definition of conditions (5.1), (5.2), acted like an 'equivalence connective' what we meant was that it was a connective such that S and S' were similarly equivalential in virtue of $\{p \leftrightarrow q\}$ as a set of equivalence formulas.[3] If we are concerned with congruential monomodal logics, we can take '$\leftrightarrow$' to be the material biconditional – the material biconditional being equivalential in all congruential monomodal logics (and hence in all normal modal logics). As before we will say that S and S' are translationally equivalent simpliciter whenever there exist appropriate translation functions τ and τ' such that S and S' are translationally equivalent via τ and τ'. One thing which is noted in Pelletier & Urquhart [2003], and which we will return to later, is that we have defined translational equivalence relative to some equivalence connective which is part of the primitive vocabulary of the logics involved. In later sections we will look at some shortcomings of this way of considering translational equivalence, and show that nonetheless it is the right sort of notion to be thinking about for considering transla-

Note that, as pointed out in fn. 1 of Smiley [1962], this use of $C(\cdot)$ is not a departure from our standard usage.

[2] As explained in Humberstone [2005b], this notion of equivalential is an adaptation of the notion explained in Czelakowski [1980], where the equivalential/non-equivalential distinction is applied instead to logics in the SET-FMLA framework (i.e. consequence relations).

[3] Caleiro & Gonçalves [2007] presents us with a relation between logics they call equipollence, formulated in terms of consequence relations, which they claim is much stronger than translational equivalence because it will apply to logics in which we cannot even define a reasonable biconditional connective. It is interesting to note that the example they give involves consequence relations according to which whenever two formulas are provably equivalent, they are also synonymous in Smiley's sense.

tional equivalence in modal logic.

We have explicitly defined translational equivalence in terms of intertranslatability in order to make it clear that all of the results which we have shown above also hold for logics which are translationally equivalent.[4] By contrast Pelletier [1984] and Pelletier & Urquhart [2003] define the notion of translational equivalence in terms of translations τ and τ' which embed S into S' and S' into S in the sense not requiring faithfulness. One fact that deserves to be pointed out at this stage is that, in this particular case, it does not matter whether we require the translations τ (and τ') to embed S into S' (or S' into S) faithfully or merely in the sense requiring them to map theorems of the source logic to theorems of the target logic – as shown in the following Theorem.

Theorem 5.0.11 (Pelletier & Urquhart [2003, p.286]). *Suppose that we have two logics* S *and* S' *and translations* τ *and* τ' *which for all formulas* A *fulfil conditions (5.1) and (5.2) in addition to the following.*

$$\begin{aligned} A \in \mathsf{S} \quad & \textit{only if} \quad \tau\,(A) \in \mathsf{S}'. \\ A \in \mathsf{S}' \quad & \textit{only if} \quad \tau'(A) \in \mathsf{S}\,. \end{aligned}$$

Then S *and* S' *are translationally equivalent via* τ *and* τ'*.*

Proof. We show only the case of S and τ. Suppose that $\tau(A) \in \mathsf{S}'$. Then by the above condition we know that $\tau'(\tau(A) \in \mathsf{S}$, which by condition (5.1) means that $A \in \mathsf{S}$, and hence that τ faithfully embeds S into S'. By the same reasoning we can also show that τ' faithfully embeds S' into S and

[4]There are a number of places in the literature when it is stated that logics which are translationally equivalent have a particular property, where the relevant proof makes no use of conditions (5.1) and (5.2). Unless close attention is paid, one might be led to think that such properties do not hold for intertranslatable logics – the example which impacted on the author is that given in Segerberg [1982] as Theorem 2.4.1 – which as we have shown above holds for merely intertranslatable logics (logics which satisfy only conditions (iii) and (iv) given in Segerberg [1982]). We will have some further comments to make on Segerberg's *syntactic equivalence* at the end of this chapter.

hence that S and S' are intertranslatable and also fulfil conditions (5.1) and (5.2) – and hence are translationally equivalent via τ and τ'. $\square$

The above theorem allows for some simplification when trying to show that two logics are translationally equivalent – all we need to do is show that certain formulas are provable within our candidate logics, rather than having to go through with the more complex arguments to do with showing that the two logics involved can be faithfully embedded into one another.

An alternative formulation of translational equivalence can be extracted from Blok & Pigozzi [1989], wherein they use a similar notion to demonstrate a particularly strong relationship between a logic (considered as a consequence relation) and a class of algebras. Let $\vdash_{\mathsf{S}}$ be a consequence relation and K a class of algebras. Then K is an *algebraic semantics for* S if there exist terms $\delta(p)$ and $\varepsilon(p)$ such that for all $B_1,\ldots,B_n,A$ we have the following.[5]

$$B_1,\ldots,B_n \vdash_{\mathsf{S}} A \iff \delta(B_1) \approx \varepsilon(B_1),\ldots,\delta(B_n) \approx \varepsilon(B_n) \models_{\mathsf{K}} \delta(A) \approx \varepsilon(A). \quad (5.3)$$

That is to say, a consequence relation has an algebraic semantics if it can be faithfully embedded into the algebraic consequence relation $\models_{\mathsf{K}}$ for some class of algebras K using a translation of the kind above.[6] K is an *equivalent*

[5]Technically this is a simplified version of what is for a logic to have an algebraic semantics. According to the official definition what is required is that there be a set of pairs of formulas in a single variable $\{\langle\delta_i(p),\varepsilon_i(p)\rangle | 1 \leq i \leq n\}$ for some $n \in Nat$, the RHS of the inset equation then being that for all $k, 1 \leq k \leq m$ we have that

$$\{\delta_i(B_j) \approx \varepsilon_i(B_j) | 1 \leq i \leq m,\ 1 \leq j \leq n\} \models_{\mathsf{K}} \delta_k(A) \approx \varepsilon_k(A).$$

This simplification does not influence any of the results mentioned, other than to make much clearer the relationship between abstract algebraic logic as described in Blok & Pigozzi [1989] and translational equivalence.

[6]One of the consequences of the above condition pointed out in Blok & Pigozzi [1989] is that if a consequence relation $\vdash$ has an algebraic semantics K, then the smallest quasi-

algebraic semantics for $\vdash_{\mathsf{S}}$ if K is an algebraic semantics for S and there is a set of equivalence formulas $\Delta(p,q)$ such that we have the following for all formulas A.

$$A \dashv\vdash_{\mathsf{S}} \Delta(\varepsilon(A), \delta(A)). \tag{5.4}$$

What we will show in the following theorem is that the appropriate analogues of these two conditions are sufficient for translational equivalence.

Theorem 5.0.12. *Suppose that* S *and* S' *are modal logics, and that* τ *and* τ' *are functions from formulas to formulas which are homonymous on* $\leftrightarrow$ *and additionally fulfil the following two conditions.*

$$\begin{array}{rc} (i) & A \in \mathsf{S} \textit{ if and only if } \tau(A) \in \mathsf{S}' \\ (ii) & A \leftrightarrow \tau(\tau'(A)) \in \mathsf{S}' \end{array}$$

Then S *and* S' *are translationally equivalent via* τ *and* τ'.

Proof. What we need to establish is that it follows from (i) and (ii) that τ' faithfully embeds S' into S and that $A \leftrightarrow \tau'(\tau(A)) \in \mathsf{S}$.

To see that τ' faithfully embeds S' into S suppose that $\tau'(A) \in \mathsf{S}$. By (i) we know this is the case if and only if $\tau(\tau'(A)) \in \mathsf{S}'$, and by (ii) that this is equivalent to $A \in \mathsf{S}'$.

To see that $A \leftrightarrow \tau'(\tau(A)) \in \mathsf{S}$ we note that $\tau(A) \leftrightarrow \tau(A) \in \mathsf{S}'$ and by (ii) that this means that $\tau(A) \leftrightarrow \tau(\tau'(\tau(A))) \in \mathsf{S}'$ and lastly by (i) that this means that $A \leftrightarrow \tau'(\tau(A)) \in \mathsf{S}$. □

This result allows us to determine the relationship between a logic and its contingency fragment. Given a normal modal logic S let us call the logic $\mathsf{S}(\nabla) = \{A | \tau_\nabla(A) \in \mathsf{S}\}$ the contingency fragment of S (denoting this logic as S^∇). Here we are taking ∇ to defined by $\nabla p =_{def} \Diamond p \wedge \Diamond \neg p$, and

variety containing K (often notated as $Q(\mathsf{K})$) is also an algebraic semantics for $\vdash$ – where a quasi-variety is the class of exactly those algebras satisfying some set of quasi-identities (universal formulas of the form $(t_1 \approx u_1 \wedge \cdots \wedge t_n \approx u_n) \to t_{n+1} \approx u_{n+1}$. More interestingly, this also means that every consequence relation with an equivalent algebraic semantics has an equivalent algebraic semantics in terms of some quasi-variety.

with τ_∇ being the translation which replaces $\Box A$ with $\nabla(A)$.[7] Following Cresswell [1988] let us say that $\Box$ *is definable (in terms of contingency) in* S iff there is a formula $C(p)$ constructed only out of the propositional variable p such that $\Box p \leftrightarrow \tau_\nabla(C(p)) \in \mathsf{S}$. Then we are able to produce the following result.

Theorem 5.0.13. *Suppose that* $\Box$ *is definable in terms of contingency in a normal modal logic* S*. Then* S *and* S^∇ *are translationally equivalent.*

Proof. It is clear from the definition of S^∇ that the translation τ_∇ faithfully embeds S^∇ into S. As $\Box$ is definable in S in terms of contingency we know that there is a formula $C(p)$ such that $\Box p \leftrightarrow \tau_\nabla(C(p)) \in \mathsf{S}$. Letting τ be the translation which replaces all occurrence of $\nabla(A)$ with $C(A)$ we are able to show, by induction upon the complexity of A – using the above result for the case where $A = \Box B$, that $A \leftrightarrow \tau_\nabla(\tau(A)) \in \mathsf{S}$, which by Theorem 5.0.12 allows us to conclude that S and S^∇ are translationally equivalent as desired. $\Box$

One of the examples of translational equivalence between monomodal logics mentioned in Pelletier & Urquhart [2003] is the following.

Theorem 5.0.14 (Lenzen [1979])**.** **S4.4** *and* **KD45** *are translationally equivalent via the translations* $\tau_{\boxdot}$ *and* $\tau_{\Diamond\Box}$*.*

What is particularly striking about the above result is that these two logics – **KD45** and **S4.4** – are both maximal logics within the range of $\tau_{\Diamond\Box}$ for **KD45**. In fact, as the following theorem shows, the relationship between translational equivalence and maximality within the range of a translation is quite strong.

Theorem 5.0.15. *Suppose that* S *and* S' *are translationally equivalent via* τ *and* τ'*. Then* S' *s maximal in the range of* τ *for* S*.*

[7] This deviates from the convention used elsewhere of having ∇ represent an arbitrary modality (as in Zolin [2000]). Whenever we are using ∇ to represent contingency rather than an arbitrary modality we will explicitly note this.

Proof. Suppose for a contradiction that there is a logic $\mathsf{S}'' \supsetneq \mathsf{S}'$ into which S can be faithfully embedded by τ. As $\mathsf{S}'' \supsetneq \mathsf{S}'$ we know there is some formula B such that $B \in \mathsf{S}''$ and $B \notin \mathsf{S}'$. Consequently by the translational equivalence of S and S' we know that $\tau'(B) \notin \mathsf{S}$. By our hypothesis this means that $\tau(\tau'(B)) \notin \mathsf{S}''$, which as $\mathsf{S}'' \supsetneq \mathsf{S}'$ means that $B \notin \mathsf{S}''$ by (5.1) giving us a contradiction. Hence S' is maximal in the range of τ for S. □

One might wonder whether we could weaken the hypothesis in the above theorem from the translational equivalence of S and S' to the mere intertranslatability of S and S'. I suspect not, but have no definitive proof of the fact – no counterexample, that is.

Translational equivalence also extends to the extensions of normal modal logics, in a particularly uniform way.

Proposition 5.0.16. *Suppose that* S *and* S' *are congruential modal logics rendered translationally equivalent by the modal-to-modal translations* τ *and* τ'. *Then for all formulas* A *and sets of formulas* Γ *we have that:*

$$(i) A \in \mathsf{S} + \Gamma \textit{ if and only if } \tau(A) \in \mathsf{S}' + \tau(\Gamma)$$
$$(ii) A \in \mathsf{S}' + \tau(\Gamma) \textit{ if and only if } \tau'(A) \in \mathsf{S} + \Gamma$$

We treat only case (i), case (ii) following similarly.

Proof. For the 'only if' direction suppose that $A \in S + \Gamma$. Then it follows that, for substitutions $\sigma_1, \ldots, \sigma_n$, and formulas $B_1, \ldots, B_n$ in Γ that:

$$(\sigma_1(B_1) \wedge \ldots \wedge \sigma_n(B_n)) \rightarrow A \in \mathsf{S}.$$

As τ is a modal-to-modal translation which faithfully embeds S into S' it follows then that $(\tau\sigma_1(B_1) \wedge \ldots \wedge \tau\sigma_n(B_n)) \rightarrow \tau(A) \in \mathsf{S}'$. So letting $\sigma' = \tau \circ \sigma$, and noting that, by Theorem 2.0.20, $\tau \circ \sigma = \sigma' \circ \tau$ it follows that:

$$(\sigma_1'(\tau(B_1)) \wedge \ldots \wedge \sigma_n'(\tau(B_n))) \rightarrow \tau(A) \in \mathsf{S}'.$$

From which it follows that $\tau(A) \in \mathsf{S}' + \tau(\Gamma)$.

For the 'if' direction suppose that $\tau(A) \in \mathsf{S}' + \tau(\Gamma)$, Then it follows that, for substitutions $\sigma_1, \ldots, \sigma_n$, and for formulas $B_1, .., B_n$ in Γ that:

$$(\sigma_1(\tau(B_1)) \wedge \ldots \wedge \sigma_n(\tau(B_n))) \to \tau(A) \in \mathsf{S}'.$$

As τ' is a modal-to-modal translation which faithfully embeds S' into S it follows that $(\tau'(\sigma_1(\tau(B_1))) \wedge \ldots \wedge \tau'(\sigma_n(\tau(B_n)))) \to \tau'(\tau(A)) \in \mathsf{S}$. So letting $\sigma' = \tau' \circ \sigma$, and noting that, by Theorem Theorem 2.0.20, $\tau' \circ \sigma = \sigma' \circ \tau'$ it follows that:

$$(\sigma'_1(\tau'(\tau(B_1))) \wedge \ldots \wedge \sigma'_n(\tau'(\tau(B_n)))) \to \tau'(\tau(A)) \in \mathsf{S}.$$

Which, by the fact that $\tau'(\tau(A)) \leftrightarrow A \in \mathsf{S}$ implies that $(\sigma'_1(B_1) \wedge \ldots \wedge \sigma'_n(B_n)) \to A \in \mathsf{S}$, from which it follows that $A \in \mathsf{S} + \Gamma$ and the result follows. □

What the above theorem tells us is that when two normal modal logics are translationally equivalent then each of their quasi-normal extensions can be faithfully embedded into a quasi-normal extension of the other. Note, though, that this does not mean that each of their quasi-normal extensions is translationally equivalent to a quasi-normal extension of the other, despite the fact that $\tau'(\tau(A)) \leftrightarrow A$ will be provable in all extensions (and hence quasi-normal extensions) of S. The reason for this is that we have no guarantee, in general, that the equivalence formulas in S will also be equivalence formulas in its extensions.

One case where we do have all the quasi-normal extensions of translationally equivalent logics being also translationally equivalent is the case of **S4.4** and **KD45** mentioned above. Recall that **S4.4** and **KD45** are rendered translationally equivalent by the translations $\tau_{\boxdot}$ and $\tau_{\Diamond\Box}$, as stated in Theorem 5.0.14. So as a direct result of the above theorem we have the following.

Corollary 5.0.17. *For all formulas A, and sets of formulas Γ we have the following.*

$$A \in \mathbf{KD45} + \Gamma \textit{ if and only if } \tau_{\Diamond\Box}(A) \in \mathbf{S4.4} + \tau_{\Diamond\Box}(\Gamma).$$
$$A \in \mathbf{S4.4} + \Gamma \textit{ if and only if } \tau_{\boxdot}(A) \in \mathbf{KD45} + \tau_{\boxdot}(\Gamma).$$

Moreover we know that all extensions of **KD45** and **S4.4** are normal (Lemma 4.2.11 and Lemma 4.2.16), and thus that $\{p \leftrightarrow q\}$ is a set of equivalence formulas in all of their extensions, allowing us to conclude the following.

Theorem 5.0.18. *For all sets of formulas* Γ *we have the following.*
(i) $\mathbf{KD45} \oplus \Gamma$ *and* $\mathbf{S4.4} \oplus \tau_{\Diamond\Box}(\Gamma)$ *are rendered translationally equivalent by the translations* $\tau_{\Diamond\Box}$ *and* $\tau_{\boxdot}$.
(ii) $\mathbf{S4.4} \oplus \Gamma$ *and* $\mathbf{KD45} \oplus \tau_{\boxdot}(\Gamma)$ *are rendered translationally equivalent by the translations* $\tau_{\boxdot}$ *and* $\tau_{\Diamond\Box}$.

We treat only case (i), case (ii) following similarly.

Proof. We begin by showing that $\mathbf{KD45} \oplus \Gamma$ and $\mathbf{S4.4} \oplus \tau_{\Diamond\Box}(\Gamma)$ are intertranslatable. By Corollary 5.0.17 and the fact that all extensions of **KD45** are normal we know that $\tau_{\Diamond\Box}$ faithfully embeds $\mathbf{KD45} \oplus \Gamma$ into $\mathbf{S4.4} \oplus \tau_{\Diamond\Box}(\Gamma)$, which is faithfully embedded into $\mathbf{KD45} \oplus \tau_{\boxdot}(\tau_{\Diamond\Box}(\Gamma))$ by $\tau_{\boxdot}$. As $\tau_{\boxdot}(\tau_{\Diamond\Box}(A)) \leftrightarrow A \in \mathbf{KD45}$ it follows that $\mathbf{KD45} \oplus \tau_{\boxdot}(\tau_{\Diamond\Box}(\Gamma)) = \mathbf{KD45} \oplus \Gamma$. So it follows that $\mathbf{KD45} \oplus \Gamma$ and $\mathbf{S4.4} \oplus \tau_{\Diamond\Box}(\Gamma)$ are intertranslatable.

That they are translationally equivalent follows from Theorem 5.0.12 as $\tau_{\boxdot}(\tau_{\Diamond\Box}(A)) \leftrightarrow A \in \mathbf{KD45}$ and $\{p \leftrightarrow q\}$ is a set of equivalence formulas in all extensions of **KD45** and **S4.4**.

$\Box$

5.1 Examples of Translational Equivalence in Modal Logic

We will now go through some examples of translational equivalence in modal logic in detail. We will begin by considering the two post-complete normal modal logics **KT!** and **KVer**. As is well known, these two modal logics are determined by the singleton reflexive, and the singleton irreflexive frame respectively. Consider now the following two modal-to-modal translations.

$$\tau_{\mathbf{KT!}}(\Box A) = \tau_{\mathbf{KT!}}(A). \qquad \tau_{\mathbf{KVer}}(\Box A) = \tau_{\mathbf{KVer}}(A) \to \tau_{\mathbf{KVer}}(A).$$

Proposition 5.1.1. *For all formulas A we have the following:*

$$\begin{array}{ll} (i) & A \in \mathbf{KT!} \textit{ only if } \tau_{\mathbf{KT!}}(A) \in \mathbf{KVer} \\ (ii) & A \in \mathbf{KVer} \textit{ only if } \tau_{\mathbf{KVer}}(A) \in \mathbf{KT!} \end{array}$$

Proof. By induction upon the length of derivations of A. □

Proposition 5.1.2. *For all formulas A we have the following:*

$$\begin{array}{ll} (i) & A \leftrightarrow \tau_{\mathbf{KVer}}(\tau_{\mathbf{KT!}}(A)) \in \mathbf{KT!} \\ (ii) & A \leftrightarrow \tau_{\mathbf{KT!}}(\tau_{\mathbf{KVer}}(A)) \in \mathbf{KVer}. \end{array}$$

Proof. From the fact that $\tau_{\mathbf{KVer}}(\tau_{\mathbf{KT!}}(A)) = \tau_{\mathbf{KT!}}(A)$ and $\tau_{\mathbf{KT!}}(\tau_{\mathbf{KVer}}(A)) = \tau_{\mathbf{KVer}}(A)$. □

Theorem 5.1.3. **KT!** *and* **KVer** *are translationally equivalent via the translations $\tau_{\mathbf{KT!}}$ and $\tau_{\mathbf{KVer}}$.*

It shouldn't be entirely surprising that **KT!** and **KVer** are translationally equivalent, as both of them can just be seen as just 'classical logic in disguise' – the □ in **KT!** just being an identity operator, and the □ in **KVer** just being a one-place version of a truth constant (hence the name **KVer** for Verum). What we will do now, though, is give an example of a somewhat more interesting case of translational equivalence.

5.1.1 Example: KD! and KTAlt$_2$ are Translationally Equivalent

Here we will show that the normal monomodal logics **KD!** and **KTAlt**$_2$ are translationally equivalent. Recall that **Alt**$_2$ is the following formula.

$$\mathbf{Alt}_2 : \quad \Box p \vee \Box(p \to q) \vee \Box(p \wedge q \to r).$$

It is easy to see that $\mathbf{KTAlt}_2$ is determined by the class of reflexive, 2-alternative frames, where a frame $\langle W, R\rangle$ is *2-alternative* whenever $|R(x)| \leq 2$ for all $x \in W$. This logic may at first glance appear to be a somewhat arbitrary one, with no natural interpretation. As it turns out though, this is none other than the logic of 'today and tomorrow' discussed by A.N. Prior and K. Segerberg (Byrd [1980]). If we consider the points in W as days, then we can interpret the truth of $\Box A$ at x as meaning that A is true at day x and the day after day x – hence the title of Segerberg [1967].

Let us say that a class of frames $\langle W, R\rangle$ is functional if $\forall x \forall y \forall z((Rxy \wedge Rxz) \to y = z)$, and totally functional if it is both functional and serial. Then **KD!** is determined by the class of totally functional frames (Segerberg [1986]). Thus for a totally functional frame $\langle W, R\rangle$ and an element $x \in W$, we can see that $R(x) = \{y\}$ for some element $y \in W$. Consequently we may occasionally call this point y the 'successor' of x. **KD!** also receives a natural tense logical interpretation when we think of the elements of W as days. In this case, we can think of the truth of $\Box A$ at a point x as meaning that A is true on the day after x.

Here we will be using the translations $\tau_{\boxdot}$ and $\tau_{\boxed{\star}}$, where $\tau_{\boxdot}$ is a familiar translation in the provability logic literature (Shavrukov [1991], Boolos [1993], Litak [2007]) where the modal function $\Box p \wedge p$ is occasionally written as $\boxdot p$. The translation $\tau_{\boxed{\star}}$ is the modal-to-modal translation which replaces all occurrences of $\Box B$ in a formula with $\boxed{\star}\,\tau_{\boxed{\star}}(B)$, where $\boxed{\star}$ is the following modal function.[8]

$$\boxed{\star}A =_{Def} \Box A \vee (\Diamond A \wedge \neg A).$$

To see that this modal function is normal in $\mathbf{KTAlt}_2$ consider the following Lemma, which shows that a formula of the form $\boxed{\star}A$ is true at a point in a reflexive, 2-alternative model exactly when A is true throughout a certain set of points.

[8]These two translations are mentioned in Segerberg [1986, p.49], where the intertranslatability of **KD!** and $\mathbf{KTAlt}_2$ is mentioned without proof.

Lemma 5.1.4. *Let $\mathcal{M} = \langle W, R, V \rangle$ be a model on a reflexive, 2-alternative frame, and define the function $f : W \to W$ as follows.*

$$f(x) = \begin{cases} y & \text{if } Rxy \text{ and } y \neq x \\ x & \text{otherwise} . \end{cases}$$

Then for all formulas A, and all points $x \in W$:

$$\mathcal{M} \models_x \boxed{\star} A \text{ if and only if } \mathcal{M} \models_{f(x)} A.$$

Proof. For the 'only if' direction suppose for a reductio that $\mathcal{M} \models_x \boxed{\star} A$ and $\mathcal{M} \not\models_{f(x)} A$. By the definition of f we know that either (i) $f(x) = y$ for some point $y \in W$ or (ii) that $f(x) = x$ and that $\forall y(Rxy \to x = y$. For (i) as Rxy and $\mathcal{M} \not\models_y A$ we know that $\mathcal{M} \not\models_x \Box A$ and also that $\mathcal{M} \not\models_x \Diamond A \wedge \neg A$, and hence that $\mathcal{M} \not\models_x \boxed{\star} A$. For (ii) we know that $\mathcal{M} \not\models_x A$ and hence that $\mathcal{M} \not\models_x \Box A$, and that while $\mathcal{M} \models_x \neg A$ we also know that $\mathcal{M} \not\models_x \Diamond A$ as the only point in $R(x)$ is x itself – a point at which A is false. Hence we can see again that $\mathcal{M} \not\models_x \boxed{\star} A$.

For the 'if' direction suppose that $\mathcal{M} \models_{f(x)} A$. By the definition of f we know that either (i) $f(x) = y$ for some point $y \in W$ such that Rxy or (ii) that $f(x) = x$ and that $\forall y(Rxy \to x = y)$. For (i) this would mean that $\mathcal{M} \models_y A$. Either $\mathcal{M} \models_x A$ or $\mathcal{M} \not\models_x A$ – the first making $\Box A$ true at x, and the second making $\Diamond A \wedge \neg A$ true at x. Consequently we can see that $\mathcal{M} \models_x \boxed{\star} A$. For (ii) this means that $\mathcal{M} \models_x A$ and that as this is the only point which is R-accessible to x that $\mathcal{M} \models_x \Box A$ and hence that $\mathcal{M} \models_x \boxed{\star} A$. □

It bears mentioning that the function f mentioned above is indeed well defined, as on a reflexive, 2-alternative frame for a given $x \in W$ there can be at most one point $y \in W$ such that $f(x) = y$. To show that **KD!** and **KTAlt**$_2$ are intertranslatable (with an aim to then showing that they are translationally equivalent), we will also need the following model construction which takes a model and converts its accessibility relation to its reflexive closure – making all points reflexive.

Definition 5.1.5. Given a model $\mathcal{M} = \langle W, R, V\rangle$ construct a new model $\mathcal{M}^\circ = \langle W, R^\circ, V\rangle$ where $R^\circ xy \iff Rxy \vee x = y$.

Theorem 5.1.6. *Suppose that $\mathcal{M} = \langle W, R, V\rangle$ is a model on a totally functional frame. Then for all formulas A and all points $x \in W$ we have that:*

$$\mathcal{M} \models_x \tau_{\boxdot}(A) \textit{ if and only if } \mathcal{M}^\circ \models_x A.$$

Proof. By induction upon the complexity of A, the only case of interest being that where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that $\mathcal{M} \models_x \Box\tau_{\boxdot}(B) \wedge \tau_{\boxdot}(B)$. Then the point y accessible from x is such that $\mathcal{M} \models_y \tau_{\boxdot}(B)$. By the inductive hypothesis then we know that $\mathcal{M}^\circ \models_x B$ and also that $\mathcal{M}^\circ \models_y B$. As $R^\circ(x) = \{x, y\}$ we can conclude that $\mathcal{M}^\circ \models_x \Box B$ as desired.

For the 'if' direction suppose that $\mathcal{M}^\circ \models_x \Box B$. Then for all points y such that $R^\circ xy$ we know that $\mathcal{M}^\circ \models_y B$. By the inductive hypothesis this means that $\mathcal{M} \models_y \tau_{\boxdot}(B)$ for all such points y. As $R(x) \subseteq R^\circ(x)$ we know that $\mathcal{M} \models_x \Box\tau_{\boxdot}(B)$ and as $x \in R^\circ(x)$ we can conclude that $\mathcal{M} \models_x \Box\tau_{\boxdot}(B) \wedge \tau_{\boxdot}(B)$ as desired. $\Box$

Theorem 5.1.7. *Suppose that $\mathcal{M} = \langle W, R, V\rangle$ is a model on a reflexive, 2-alternative frame, and that $\mathcal{M}^f = \langle W, f, V\rangle$ where f is as defined in Lemma 5.1.4. Then for all formulas A and all points $x \in W$ we have that:*

$$\mathcal{M} \models_x A \textit{ if and only if } \mathcal{M}^f \models_x \tau_{\boxdot}(A).$$

Proof. By induction upon the complexity of A, the only case of interest being that where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that $\mathcal{M} \models_x \Box B$. This means that for all points $y \in R(x)$ that $\mathcal{M} \models_y B$. By the inductive hypothesis this means that for all such points $y \in W$ that $\mathcal{M}^f \models_y \tau_{\boxdot}(B)$. As $f(x) \cup \{x\} = R(x)$ we know then that $\mathcal{M}^f \models_x \tau_{\boxdot}(B)$ and also that $\mathcal{M}^f \models_{f(x)} \tau_{\boxdot}(B)$. Consequently we can see that $\mathcal{M}^f \models_x \Box\tau_{\boxdot}(B) \wedge \tau_{\boxdot}(B)$.

For the 'if' direction suppose that $\mathcal{M}^f \models_x \Box\tau_{\boxdot}(B) \wedge \tau_{\boxdot}(B)$. Then we know that $\mathcal{M}^f \models_{f(x)} \tau_{\boxdot}(B)$, and by the inductive hypothesis that $\mathcal{M} \models_x B$

and $\mathcal{M} \models_{f(x)} B$. As $f(x) \cup \{x\} = R(x)$ we can see that this means that for all $y \in R(x)$ we have that $\mathcal{M} \models_y B$ and thus that $\mathcal{M} \models_x \Box B$ as desired. $\square$

Theorem 5.1.8. $\tau_{\boxdot}$ *faithfully embeds* $\mathbf{KTAlt}_2$ *into* $\mathbf{KD!}$.

Proof. What we want to show is that, for all formulas A we have:

$$A \in \mathbf{KTAlt}_2 \text{ if and only if } \tau_{\boxdot}(A) \in \mathbf{KD!}.$$

For the 'only if' direction suppose that $\tau_{\boxdot}(A) \notin \mathbf{KD!}$. Then there is a totally functional model $\mathcal{M} = \langle W, R, V \rangle$ and a point $x \in W$ such that $\mathcal{M} \models_x \tau_{\boxdot}(B)$. By Theorem 5.1.7 we thus know that $\mathcal{M}^\circ \not\models_x A$ and as $\mathcal{M}^\circ$ is reflexive and 2-Alternative we thus know that $A \notin \mathbf{KTAlt}_2$.

For the 'if' direction suppose that $A \notin \mathbf{KTAlt}_2$. Then there is a reflexive, 2-Alternative model $\mathcal{M} = \langle W, R, V \rangle$ and a point $x \in W$ such that $\mathcal{M} \not\models_x A$. By Theorem 5.1.6 we know that $\mathcal{M}^f \not\models_x \tau_{\boxdot}(A)$ and hence – as this model is clearly a totally functional one – that $\tau_{\boxdot}(A) \notin \mathbf{KD!}$. $\square$

Theorem 5.1.9. *Suppose that* $\mathcal{M} = \langle W, R, V \rangle$ *is a model on a reflexive, 2-alternative frame, and that* $\mathcal{M}^f = \langle W, f, V \rangle$ *where* f *is as in Lemma 5.1.4. Then for all formulas A and all* $x \in W$ *we have that:*

$$\mathcal{M} \models_x \tau_{\boxed{\star}}(A) \textit{ if and only if } \mathcal{M}^f \models_x A.$$

Proof. By induction upon the complexity of A, the only case of interest being that where $A = \Box B$ for some formula B – which follows by Lemma 5.1.4. $\square$

Theorem 5.1.10. *Suppose that* $\mathcal{M} = \langle W, R, V \rangle$ *is a model on a totally functional frame. Then for all formulas A and all points* $x \in W$ *we have that:*

$$\mathcal{M} \models_x A \textit{ if and only if } \mathcal{M}^\circ \models_x \tau_{\boxed{\star}}(A).$$

Proof. By induction upon the complexity of A, the only case of interest being that where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that $\mathcal{M} \models_x \Box B$. Then for the unique successor of x – the point y – we know that $\mathcal{M} \models_x B$. By the inductive hypothesis this means that $\mathcal{M}^\circ \models_y \tau_{\boxed{\star}}(B)$. We also know that $f(x) = y$ and hence by Lemma 5.1.4 that $\mathcal{M}^\circ \models_x \boxed{\star}\tau_{\boxed{\star}}(B)$.

For the 'if' direction suppose that $\mathcal{M} \not\models_x \Box B$. Then there is a point $y \in R(x)$ such that $\mathcal{M} \not\models_y B$. By the inductive hypothesis this means that $\mathcal{M}^\circ \not\models_y \tau_{\boxed{\star}}(B)$. As Rxy we know that $R^\circ xy$ and thus that $\mathcal{M}^\circ \not\models_x \Box\tau_{\boxed{\star}}(B)$. As $R^\circ(x) = \{x, y\}$ we can also see that $\mathcal{M}^\circ \not\models_x \Diamond\tau_{\boxed{\star}}(B) \wedge \neg\tau_{\boxed{\star}}(B)$. Consequently, $\mathcal{M}^\circ \not\models_x \Box\tau_{\boxed{\star}}(B) \vee (\Diamond\tau_{\boxed{\star}}(B) \wedge \neg\tau_{\boxed{\star}}(B))$. □

Theorem 5.1.11. $\tau_{\boxed{\star}}$ *faithfully embeds* **KD!** *into* $\mathbf{KTAlt}_2$.

Proof. What we want to show is that, for all formulas A we have that:

$$A \in \mathbf{KD!} \text{ if and only if } \tau_{\boxed{\star}}(A) \in \mathbf{KTAlt}_2.$$

For the 'only if' direction suppose that $\tau_{\boxed{\star}}(A) \notin \mathbf{KTAlt}_2$. Then there is a reflexive, 2-alternative model $\mathcal{M} = \langle W, R, V \rangle$ and a point $x \in W$ such that $\mathcal{M} \not\models_x \tau_{\boxed{\star}}(A)$. By Theorem 5.1.9 we thus know that $\mathcal{M}^f \not\models_x A$. As $\mathcal{M}^f$ is a totally functional model we thus know that $A \notin \mathbf{KD!}$.

For the 'if' direction suppose that $A \notin \mathbf{KD!}$. Then there is a functional model $\mathcal{M} = \langle W, R, V \rangle$ and a point $x \in W$ such that $\mathcal{M} \not\models_x A$. By Theorem 5.1.10 we know that $\mathcal{M}^\circ \not\models_x \tau_{\boxed{\star}}(A)$, and hence – as this model is clearly on a reflexive, 2-alternative frame – that $\tau_{\boxed{\star}}(A) \notin \mathbf{KTAlt}_2$. □

So far we have managed to show that $\mathbf{KTAlt}_2$ and **KD!** are intertranslatable, all that remains to be shown then is that the translations $\tau_{\boxdot}$ and $\tau_{\boxed{\star}}$ are inverses in the sense that if we take a formula in either logic, and translate it into the other using the appropriate translation, and then translate it back into the starting logic, we then end up with a formula equivalent to what we started out with. That is to say, we now need to show that conditions (5.1) and (5.2) on page 93 are fulfilled.

Firstly we will begin by showing that $\Box A \leftrightarrow \tau_{\boxast}(\tau_{\boxdot}(\Box A)) \in \mathbf{KTAlt}_2$.

(1)	$\Box p \leftrightarrow (\Box p \wedge p)$	$\mathbf{T}$
(2)	$\Box p \leftrightarrow (\Box p \wedge p) \vee (\Diamond p \wedge \neg p)$	(1), $\mathbf{Alt}_2$
(3)	$\Box p \leftrightarrow (\Box p \vee (\Diamond p \wedge \neg p)) \wedge p$	(2), TF

Next, we show $\Box A \leftrightarrow \tau_{\boxdot}(\tau_{\boxast}(\Box A)) \in \mathbf{KD!}$.

(1)	$\Box p \leftrightarrow (\Box p \wedge p) \vee (\Box p \wedge \neg p)$	TF
(2)	$\Box p \leftrightarrow (\Box p \wedge p) \vee (\Diamond p \wedge \neg p)$	(1), $\mathbf{D!}$
(3)	$\Box p \leftrightarrow (\Box p \wedge p) \vee ((\Diamond p \wedge \neg p) \vee (p \wedge \neg p)$	(2), TF
(4)	$\Box p \leftrightarrow (\Box p \wedge p) \vee ((\Diamond p \vee p) \wedge \neg p)$	(3), TF

Given the above two results it is easy to prove by induction upon the complexity of formulas that $A \leftrightarrow \tau_{\boxast}(\tau_{\boxdot}(A)) \in \mathbf{KTAlt}_2$ (or equally $A \leftrightarrow \tau_{\boxdot}(\tau_{\boxast}(A)) \in \mathbf{KD!}$), allowing us to conclude the following via Theorem 5.0.12.

Theorem 5.1.12. $\mathbf{KTAlt}_2$ *and* **KD!** *are translationally equivalent via the translations* $\tau_{\boxdot}$ *and* $\tau_{\boxast}$.

An alternative axiomatization of **KD!** to the one given above (where we take the normal extension of **K** by the axiom $\Box p \leftrightarrow \Diamond p$) is to take **KD!** to be $\mathbf{KDAlt}_1$. The above theorem can thus be seen as showing that the logics $\mathbf{KTAlt}_2$ and $\mathbf{KDAlt}_1$ are translationally equivalent. One might wonder we can generalize this result to showing that, for all $n \in Nat$, $\mathbf{KDAlt}_n$ and $\mathbf{KTAlt}_{n+1}$ are translationally equivalent via $\tau_{\boxdot}$ and $\tau_{\boxast}$.

Firstly we will consider whether, for $n \geq 2$, $\tau_{\boxast}$ faithfully embeds $\mathbf{KDAlt}_n$ into $\mathbf{KTAlt}_{n+1}$ – the $n = 1$ case having been covered above. As it happens $\tau_{\boxast}$ does not even embed $\mathbf{KDAlt}_n$ into $\mathbf{KTAlt}_{n+1}$ in the sense requiring that the translations of theorems of $\mathbf{KDAlt}_n$ be theorems of $\mathbf{KTAlt}_{n+1}$. To see this consider the $\tau_{\boxast}$ translation of **K**.

$$\tau_{\boxast}(\mathbf{K}): \quad (\Box(p \to q) \vee (\Diamond(p \to q) \wedge \neg(p \to q))) \to ((\Box p \vee (\Diamond p \wedge \neg p)) \to (\Box q \vee (\Diamond q \wedge \neg q)))$$

It is easy to show that $\tau_{\boxed{\star}}(\mathbf{K})$ isn't a theorem of $\mathbf{KTAlt}_{n+1}$ for $n > 1$, for consider the following model.

- $W = \{w, x_1, x_2, \ldots, x_n\}$.
- $R = \{\langle w, x_i\rangle | 1 \leq i \leq n\} \cup \{\langle x, x\rangle | x \in W\}$
- $V(p) = \{x_1\}$, $V(q) = \{w, x_1\}$.

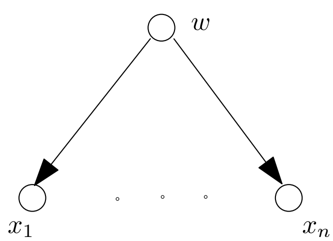

Figure 5.1: A countermodel to the provability of $\tau_{\boxed{\star}}(\mathbf{K})$ in $\mathbf{KTAlt}_{n+1}$, the open circles representing reflexive points.

It is easy to see that '$\Box(p \to q)$' is true at w in the above model, and that '$\Diamond p \wedge \neg p$' is also true at w, but that '$\Box q \vee (\Diamond q \wedge \neg q)$' is false at w – q being false at x_2 (thus making $\Box q$ false at w) and also that q is true at w (thus making $\Diamond q \wedge \neg q$ false at w). Thus we can see that the modal function $\boxed{\star}$ isn't even normal in the modal logics $\mathbf{KTAlt}_{n+1}$ for $n \geq 2$. This raises the following obvious open question.

Open Problem 5.1.13. Is there a translation τ which faithfully embeds $\mathbf{KDAlt}_n$ into $\mathbf{KTAlt}_{n+1}$ for $n > 1$?

By contrast, we are able to show that, for all $n \in Nat$, $A \in \mathbf{KTAlt}_{n+1}$ if and only if $\tau_{\boxdot}(A) \in \mathbf{KDAlt}_n$. To show this we will require the following model construction – which takes a model on a reflexive frame and converts it to one where a point $x \in W$ is reflexive in the new frame iff its only alternative is itself in the original frame.

Definition 5.1.14. Given a model $\mathcal{M} = \langle W, R, V\rangle$ define a new model $\mathcal{M}^- = \langle W, R^-, V\rangle$ where $R^-xy \iff (Rxy \wedge x \neq y) \vee (x = y \wedge \forall z(Rxz \to y = z))$.

Theorem 5.1.15. *Suppose that $\mathcal{M} = \langle W, R, V\rangle$ is a model on a serial n-Alternative frame. Then for all formulas A and all points $x \in W$ we have that:*

$$\mathcal{M} \models_x \tau_{\boxdot}(A) \textit{ if and only if } \mathcal{M}^\circ \models_x A.$$

Proof. By induction upon the complexity of A, the only case of interest being that where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that $\mathcal{M} \models_x \Box\tau_{\boxdot}(B) \wedge \tau_{\boxdot}(B)$. Then all points y such that Rxy are such that $\mathcal{M} \models_y \tau_{\boxdot}(B)$. By the inductive hypothesis we know that $\mathcal{M}^\circ \models_x B$, and also that $\mathcal{M} \models_y B$ for all y such that Rxy. As $R^\circ(x) = R(x) \cup \{x\}$ it follows that B is true at all the points which are R°-accessible to x in $\mathcal{M}^+$ and thus that $\mathcal{M}^\circ \models_x \Box B$.

For the 'if' direction suppose that $\mathcal{M}^\circ \models_x \Box B$. Then for all points y such that $R^\circ xy$ we know that $\mathcal{M}^\circ \models_y B$. By the inductive hypothesis this means that $\mathcal{M} \models_y \tau_{\boxdot}(B)$ for all such points y. As $R(x) \subseteq R^\circ(x)$ we know that $\mathcal{M} \models_x \Box\tau_{\boxdot}(B)$ and as $x \in R^\circ(x)$ we can conclude that $\mathcal{M} \models_x \Box\tau_{\boxdot}(B) \wedge \tau_{\boxdot}(B)$ as desired. □

Theorem 5.1.16. *Suppose that $\mathcal{M} = \langle W, R, V\rangle$ is a reflexive n-Alternative frame. Then for all formulas A and all points $x \in W$ we have that:*

$$\mathcal{M} \models_x A \textit{ if and only if } \mathcal{M}^- \models_x \tau_{\boxdot}(A). \tag{5.5}$$

Proof. By induction upon the complexity of A, the only case of interest being that where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that $\mathcal{M} \models_x \Box B$. Then for all points y such that Rxy we know that $\mathcal{M} \models_y B$. By the inductive hypothesis this means that $\mathcal{M}^- \models_y \tau_{\boxdot}(B)$. As $R^-(x) \subseteq R(x)$ we know that this means that $\mathcal{M}^- \models_x \Box\tau_{\boxdot}(B)$. By the reflexivity of R it follows that $\mathcal{M} \models_x B$ and thus by the inductive hypothesis that $\mathcal{M}^- \models_x \tau_{\boxdot}(B)$. Consequently $\mathcal{M}^- \models_x \Box\tau_{\boxdot}(B) \wedge \tau_{\boxdot}(B)$.

For the 'if' direction suppose that $\mathcal{M}^- \models_x \Box\tau_{\boxdot}(B) \wedge \tau_{\boxdot}(B)$. Then for all points y such that R^-xy we have $\mathcal{M}^- \models_y \tau_{\boxdot}(B)$. By the inductive hypothesis it follows that $\mathcal{M} \models_x B$ and for all points y such that R^-xy that $\mathcal{M} \models_y B$. As $R(x) = R^-(x) \cup \{x\}$ it follows that $\mathcal{M} \models_x \Box B$ as desired. □

Theorem 5.1.17. *For all $n \in Nat$ and all formulas A we have the following:*

$$A \in \mathbf{KTAlt}_{n+1} \text{ if and only if } \tau_{\boxdot}(A) \in \mathbf{KDAlt}_n. \tag{5.6}$$

Proof. For the 'only if' direction suppose that $\tau_{\boxdot}(A) \notin \mathbf{KDAlt}_n$. Then there is a model on a serial n-Alternative frame $\mathcal{M} = \langle W, R, V\rangle$ and a point $x \in W$ such that $\mathcal{M} \not\models_x \tau_{\boxdot}(A)$. By Theorem 5.1.15 we have that $\mathcal{M}^+ \not\models_x A$. As this is clearly a model on a reflexive $n+1$-Alternative frame it follows that $A \notin \mathbf{KTAlt}_{n+1}$.

For the 'if' direction suppose that $A \notin \mathbf{KTAlt}_{n+1}$. Then there is a model on a reflexive $n+1$-alternative frame $\mathcal{M} = \langle W, R, V\rangle$ and a point $x \in W$ such that $\mathcal{M} \not\models_x A$. By Theorem 5.1.16 we have that $\mathcal{M}^- \not\models_x \tau_{\boxdot}(A)$. As $\langle W, R^-\rangle$ is clearly a serial, n-Alternative frame we can conclude that $\tau_{\boxdot}(A) \notin \mathbf{KTAlt}_{n+1}$.

□

5.2 Is Intertranslatability Translational Equivalence?

As we mentioned at the start of this chapter in fn. 4, there are instances in the literature where people attribute properties to translational equivalence which hold because the logics are intertranslatable – no special appeals to translational equivalence being needed. One possible explanation for this would be if there was no difference between translational equivalence and intertranslatability. This would amount to showing that whenever a pair of logics S and S' are intertranslatable then they are translationally equivalent – the other direction following from Theorem 5.0.11.

It is this question, of whether intertranslatability is translational equivalence, which we will address in this section. Before doing so, though, we will find it useful to briefly talk about invariant conditions for translational equivalence.

5.2.1 Invariant Conditions

A property P is an *invariant for translational equivalence* if whenever S and S' are translationally equivalent then either both have P or neither of them does. Invariant conditions are very good for telling us when two logics are not translationally equivalent, and it is to this work they are put in Pelletier & Urquhart [2003]. The most useful invariant condition they propose is the following.

Theorem 5.2.1 (Pelletier & Urquhart [2003, p.278]). *If* S *and* S' *are two normal modal logics differing in the number of finite relational frames in a given cardinality validating them, then they are not translationally equivalent.*

This result allows us to show that the majority of the common normal modal logics are not translationally equivalent to each other. A very useful necessary condition for a pair of modal logics to be translationally equivalent follows from this result – the *cardinality criterion*. If S and S' are translationally equivalent normal modal logics then they are both validated by the same number of finite frames of size n, for all $n \in Nat$. This leads us to the obvious question mentioned in Pelletier & Urquhart [2003] of whether this condition is not only a necessary condition, but also a sufficient one for translational equivalence. What we will show now is that this is not the case.

Following Boolos [1993], let us say that a relation R is *converse wellfounded* if for every nonempty set X, there is an R-greatest element of X, and element w of X such that Rwx for no $x \in W$.

Theorem 5.2.2 (van Benthem [1983, p.47]). **GL** *is valid on* $\langle W, R\rangle$ *iff* R *is transitive and converse wellfounded.*

A relation R is *converse weakly wellfounded* if for every nonempty set X, there is an R-greatest element of X, an element w of X such that Rwx for no $x \in W$ other than w itself.

Theorem 5.2.3 (van Benthem [1983, p.48]). **Grz** *is valid on* $\langle W, R\rangle$ *iff* R *is transitive, reflexive and converse weakly wellfounded.*

It is easy to see that the reflexive closure of a transitive and converse wellfounded frame is a transitive, reflexive and converse weakly wellfounded frame. Likewise that the irreflexivization of a transitive, reflexive and converse weakly wellfounded frame is a transitive and converse wellfounded one, giving us the following result. When $\langle W, R\rangle$ is a frame, let $\langle W, R\rangle^{\bullet}$ be the frame $\langle W, R^{\bullet}\rangle$ where $R^{\bullet} = R \setminus \{\langle x, x\rangle | \langle x, x\rangle \in R\}$, and recall that $\langle W, R\rangle^{\circ}$ is the frame $\langle W, R^{\circ}\rangle$ where $R^{\circ} = R \cup \{\langle x, x\rangle | x \in W\}$.

Theorem 5.2.4. *Suppose that* $\langle W, R\rangle$ *is a finite frame. Then:*
(i) $\langle W, R\rangle^{\bullet}$ *is a frame for* **GL** *iff* $\langle W, R\rangle^{\circ}$ *is a frame for* **Grz**.
(ii) $\langle W, R\rangle^{\circ}$ *is a frame for* **Grz** *iff* $\langle W, R\rangle^{\bullet}$ *is a frame for* **GL**.

Proof. Follows from the fact the the finite frames for **GL** and **Grz** are the unions of finite irreflexive and transitive trees, a and finite reflexive and transitive trees, and the fact that $(\cdot)^{\bullet}$ and $(\cdot)^{\circ}$ are 1-1 on such frames. □

Corollary 5.2.5. **GL** *and* **Grz** *are valid on the same number of frames of size* n*, for all* $n \in Nat$.

It is well known that **Grz** can be faithfully embedded into **GL** by the translation $\tau_{\boxdot}$ ([Boolos 1980, p.15]). What is perhaps less well known is that, as shown in Boolos [1980], there can be no modal-to-modal translation which faithfully embeds **GL** into **Grz** – a proof of which we give here.

Lemma 5.2.6. *Let* S *be an extension of* **KD**. *Then for all unary contexts* $C(p)$ *we have that either* $C(\bot) \in \mathsf{S}$ *or* $\neg C(\bot) \in \mathsf{S}$.

Theorem 5.2.7. *Suppose that* S *is a modal logic such that* $\Box\bot \notin \mathsf{S}$ *and* $\neg\Box\bot \notin \mathsf{S}$. *Then* S *cannot be faithfully embedded by a modal-to-modal translation into any* S' *such that* $\mathsf{S}' \supseteq \mathbf{KD}$.

Proof. Suppose that τ is a modal-to-modal translation such that $\tau(\Box A) = C(\tau(A))$. Then, as $\mathsf{S}' \supseteq \mathbf{KD}$ by Lemma 5.2.6 we know that either $C(\bot) \in \mathsf{S}'$ or $\neg C(\bot) \in \mathsf{S}'$. But S proves neither $\Box\bot$ nor $\neg\Box\bot$, and thus the translation fails in the 'only if' direction. $\Box$

Corollary 5.2.8. **GL** *cannot be faithfully embedded into* **Grz**.

The above results now put us in a position to answer the question mentioned above, which happens to be the following open problem from Pelletier & Urquhart [2003].

> PROBLEM 7. Are there pairs of normal modal logics S and S' such that S and S' have the same number of frames of size n, for all $n \in Nat$, but S and S' are not translationally equivalent?

The above results show that **GL** and **Grz** constitute such a pair of logics, having the same number of frames of size n, for all n, but failing to be intertranslatable (and hence also failing to be translationally equivalent). Thus we can see that this condition is a necessary, but not sufficient condition for a pair of logics to be translationally equivalent.

We return now to the question with which we started this section, namely, whether translational equivalence is just intertranslatability. Counterexamples like this have been present in the extant literature for quite some time – the earliest being present in the thesis of R. Montague (reported in de Bouvère [1965]) – all of them showing that this is not the case for a variety of different logical frameworks. For example, in Wójcicki [1984] it is shown that this is not the case for arbitrary consequence relations, his example involving consequence relations over the propositional language consisting of a single unary operator.

Considerations of the distinction between the analogues of intertranslatability and translational equivalence between first order theories have been around for some time – Corcoran [1980] being a survey of such considerations, which are also touched on in section 5 of Humberstone [2005a]. In Corcoran [1983] we are given a general method for, given a consistent first-order theory T, finding a theory ET (the 'elongation' of T) such that T and ET are intertranslatable but not translationally equivalent.[9] Assume that T is the theory of a single binary relation R. Then ET results from replacing every occurrence of a subformula of the form Rxy in T with $ERxyy$. Corcoran then has a general theorem showing that ET and T are not definitionally equivalent. In Andréka *et al.* [2005] we are given an example of two first order theories which are intertranslatable but not translationally equivalent.

Translational equivalence between first order theories is especially interesting in light of its connection to modal logic, which allows us to make a point which will be of some relevance in a moment. Let *Ref* and *Irr* be the first order theories of a reflexive binary relation R, and irreflexive binary relation $R^\bullet$ respectively. Then it is easy to see that the following translations render *Ref* and *Irr* intertranslatable.[10]

$$\begin{aligned} Rxy &=_{df} R^\bullet xy \vee x = y. \\ R^\bullet xy &=_{df} Rxy \wedge x \neq y. \end{aligned}$$

It is likewise easy to see that $Rxy \leftrightarrow (Rxy \wedge x \neq y) \vee x = y$ is a theorem of *Ref*, and hence by the appropriate analogue of Theorem 5.0.12, that these two theories are translationally equivalent. By contrast, if we consider the restriction of the first order language (sometimes called the guarded fragment) into which the modal language can be translated, then in this context these two theories are not translationally equivalent (as it happens

[9]Specifically Corcoran shows that T and ET are not definitionally equivalent – i.e. have no common definitional extension.

[10]The following example is mentioned in fn. 29 of Humberstone [2005a].

these are even difficult to state in this context) – as if they were this would mean that the normal modal logics **KT** and **K** were translationally equivalent – and these logics are not even intertranslatable, as was shown at the start of chapter 3. This is a good example of the degree to which translational equivalence, like most translation phenomena, is heavily dependent on the expressive resources available to us in the object language.

We return now to the question of particular relevance to us, namely whether in modal logic translational equivalence is just intertranslatability. In Pelletier & Urquhart [2008], correcting a putative counterexample in Pelletier & Urquhart [2003], a pair of logics are given which are shown to be intertranslatable but not translationally equivalent – thus constituting a counterexample to this claim. The normal modal logics in question have two propositional constants I and E (initial and end-point).

Given some set $X \subseteq Nat$, the logic L_X is the smallest normal modal logic containing the following axioms.

$$\begin{array}{ll} (1) & \Diamond A \to \Box A \\ (2) & \neg(I \wedge E) \\ (3) & \neg\Diamond I \\ (4) & I \to \neg\Diamond^k E, \text{ for } k \in X. \end{array}$$

Frames for this logic are of the form $\langle W, R, I, E\rangle$ where W is a nonempty set, R a functional relation on W, and I and E are disjoint subsets of W. We can think of the set I as being the set of *initial points* and E as the set of *end points* of the frame. Generated frames for this logic can be thought of as coming in two types. The first type of generated frame $\langle W, R, I, E\rangle$ are the functional frames with $I = \emptyset$ and $E \subseteq W$ – as the only conditions we place upon the set E have to do with its interaction with I we are at liberty to choose arbitrary subsets of W for the truth set of E in these frames. The second type of frame are those generated frames $\langle W, R, I, E\rangle$ which are generated by a point $x \in W$ such that $x \notin R^*(x)$ (where R^* is the ancestral of R) for which $I = \{x\}$ and $E \subseteq \{y|\neg R^i xy \text{ for } i \in X\}$.

What Pelletier and Urquhart show can be captured by the following two theorems.

Theorem 5.2.9. *Let* $X = \{4^k | k \in Nat\}$ *and* $Y = \{2 \cdot 4^k | k \in Nat\}$. *Then the logics* L_X *and* L_Y *are intertranslatable via* $\tau_{\Box\Box}$ *and* $\tau_{\Box\Box}$.

Theorem 5.2.10. L_X *and* L_Y *differ in the number of 2-element frames validating them.*

In particular, the frame $\langle \{x,y\}, \{\langle x,y \rangle\}, \{x\}, \{y\} \rangle$ validates L_Y but not L_X. We will leave aside the details of the proof by cases here – turning instead to a methodological point.

Technically what Pelletier and Urqhart have shown is that, in normal modal logic over languages containing two or more propositional constants, intertranslatability is not translational equivalence. It is, in a sense, dishonest to think that this shows that intertranslatability is not translational equivalence in normal modal logic with no propositional constants, as propositional constants bring with them a significant amount of extra expressive power. Considering the degree to which the expressive power of the object language can impact questions of intertranslatability and translational equivalence, it is perhaps not correct to say that the above example has shown that translational equivalence is not intertranslatability in modal logic. For this to be the case, what we would like is an example involving a pair of normal modal logics – without any propositional constants – which are intertranslatable but not translationally equivalent. In the next section we will give an example of a pair of quasi-normal modal logics which are intertranslatable but not translationally equivalent, but which draw out an interesting clarification which needs to be made regarding the notion of translational equivalence.

5.2.2 The Hazen Example – Quasi-Normal Logics which are Intertranslatable but not Translationally Equivalent.

Recall that a modal logic S is quasi-normal whenever it is an extension of the smallest normal modal logic $\mathbf{K}$. In this section what we will do is provide an example, similar to that given in Pelletier & Urquhart [2008], of a pair of modal logics which are intertranslatable but not translationally equivalent. In particular the pair of modal logics we will be interested in are two quasi-normal modal logics in the language of standard monomodal logic.

That the two logics mentioned below are intertranslatable was suggested by Allen Hazen in private correspondence. Throughout this section we will let i_n be an abbreviation for 2^{2n}, and j_n be an abbreviation for $2^{(2n)+1}$. The logics with which we will be concerned are the quasi-normal modal logics S_i and S_j described below.

$$\begin{aligned} \mathsf{S}_i &: \mathbf{K} + \{\Diamond^{i_n} p \to \Box^{i_n} p : n \in Nat\}. \\ \mathsf{S}_j &: \mathbf{K} + \{\Diamond^{j_n} p \to \Box^{j_n} p : n \in Nat\}. \end{aligned}$$

These two logics are characterized by (point generated) frames with distinguished elements which have 'wasp-waists' at a certain number of R-steps away from the generating designated element – every i_n or j_n R-steps respectively. That is to say, the frames for these logics fulfil the following 'wasp-waist' condition for every $n \in Nat$. Let $\mathfrak{F} = \langle W, R, \{x\}\rangle$ be a frame with distinguished element x (frame w.d.e. for short).[11] Then $\textsc{wasp-waist}_m$ is the following first order condition on frames w.d.e.

$$\textsc{wasp-waist}_m : \quad \forall y \forall z (R^m xy \Rightarrow (R^m xz \Rightarrow z = y)). \tag{5.7}$$

[11] A frame w.d.e. differs from a standard Kripke frame in that a formula is considered to be valid on that frame whenever it is true at the distinguished element on all models on that frame.

Proposition 5.2.11. *(i)* S_i *is determined by the class of all frames w.d.e. which satisfy* WASP-WAIST$_{i_n}$ *for all* $n \in Nat$. *(ii)* S_j *is determined by the class of all frames w.d.e. which satisfy* WASP-WAIST$_{j_n}$ *for all* $n \in Nat$.

Theorem 5.2.12. $A \in \mathsf{S}_i$ *if and only if* $\tau_{\Box\Box}(A) \in \mathsf{S}_j$.

Proof. For the 'if' direction suppose that $A \notin \mathsf{S}_i$. Then there is a model $\mathcal{M} = \langle W, R, \{x\}, V\rangle$ such that $\mathcal{M} \not\models_x A$. So by Theorem 2.1.5 $\mathcal{M}^+ \not\models_x \tau_{\Box\Box}(A)$. All that remains to be shown then is that $\mathcal{M}^+$ is a model on a frame for S_j. Suppose that, for some $n \in Nat$ that $\Diamond^{j_n} p \to \Box^{j_n} p$ was not valid on the frame for $\mathcal{M}^+$ at x. Then there is no $y \in W^+$ such that $R^{+2^{(2n)+1}}(x) \neq \{y\}$. As $R^{+2} = R$ it follows that $(R^{2^{(2n)+1}})^2(x) \neq \{y\}$. That is to say $R^{2^{(2n)+2}}(x) \neq \{y\}$, which is just $R^{i_{n+1}}(x) \neq \{y\}$, for any $y \in W^+$, and hence as $W \subseteq W^+$ for any $y \in W$. But $\mathcal{M}$ is a model on a frame for S_i, so this is impossible. Consequently $\mathcal{M}^+$ is a model on a frame for S_j, from which it follows that $\tau_{\Box\Box}(A) \notin \mathsf{S}_j$.

For the 'only if' direction we proceed by induction upon the length of derivations of A in S_i – axiomatized as suggested by the introduction above.[12] For the basis case, where A is an instance of an axiom, the only case of interest being that where $A = \Diamond^{i_n} B \to \Box^{i_n} B$. In this case we know that, as $\tau_{\Box\Box}(\Diamond^{i_n} p \to \Box^{i_n} p) = \Diamond^{j_n} p \to \Box^{j_n} p$ and that S_j is closed under uniform substitution that $\tau_{\Box\Box}(\Diamond^{i_n} B \to \Box^{i_n} B) \in \mathsf{S}_j$. The inductive case is routine. $\Box$

Theorem 5.2.13. $A \in \mathsf{S}_j$ *if and only if* $\tau_{\Box\Box}(A) \in \mathsf{S}_i$.

5.2.2.1 Translational Equivalence and Non-Normal Modal Logics

Suppose that S is a logic on a propositional language L_{S}, and $C(p,q)$ is an equivalence connective in S. Suppose also that O is a connective not

[12] That is to say, we are taking S_i to be axiomatized by taking $\Diamond^{i_n} p \to \Box^{i_n} p$ as an axiom for each $n \in Nat$, in addition to axioms for **K** and the rules Modus Ponens and Uniform Substitution.

in L_{S}. Then a logic S' on the language $L_{\mathsf{S}'}$ which results from adding the connective O to L_{S} is a *definitional extension of* S if it is the result of adding an axiom of the following form to S, where $A(p_1,\ldots,p_n)$ is a formula form L_{S} containing only the propositional variables $p_1,\ldots,p_n$.

$$C(O(p_1,\ldots,p_n),A(p_1,\ldots,p_n)). \tag{5.8}$$

Theorem 5.2.14 (Pelletier & Urquhart [2003]). *Two logics* S *and* S' *are translationally equivalent if and only if they have a common definitional extension.*

Now there is nothing in the above definition which requires anything other than the properties listed above regarding the behaviour of C – i.e. that it is an equivalence connective, so this result holds even for non-normal modal logics, as one would have expected. In order to show that the logics S_i and S_j defined above are not translationally equivalent we will need to use a reformulation of a result given in Pelletier & Urquhart [2003] concerning modal algebras. If $\mathfrak{A}$ is a modal algebra, and D a filter in $\mathfrak{A}$ then we will call the pair $\langle\mathfrak{A},D\rangle$ a *modal matrix*, and a *matrix for* S whenever $\langle\mathfrak{A},D\rangle\models\mathsf{S}$. Following Pelletier & Urquhart let us say that two classes of matrices F and G are *coalescent* if there is a class H of matrices which is a common definitional extension of both F and G.

Theorem 5.2.15. *If* S *and* S' *are translationally equivalent non-normal modal logics, and* M *and* M' *are the classes of modal matrices validating them, then* M *and* M' *are coalescent.*

Proof. The argument proceeds as in Pelletier & Urquhart [2003, p.273]. □

Following Zakharyaschev & Chagrov [1997], given a frame $\langle W,R\rangle$, we will let $\langle W,R\rangle^+$ denote the algebra $\langle 2^W,\cap,\cup,\supset,\emptyset,\Box\rangle$, where $\supset$ and $\Box$ are defined as follows, where $X,Y\subseteq W$.

$$\begin{aligned} X\supset Y &= (W\setminus X)\cup Y \\ \Box X &= \{x\in W:\forall y(Rxy\to y\in X)\}. \end{aligned}$$

Given a frame with distinguished elements $\langle W, R, D\rangle$ where D is the set of distinguished elements in the frame, define the *dual of* $\langle W, R, D\rangle$ ($\langle W, R, D\rangle^+$) to be the matrix $\langle\langle W, R\rangle^+, D^+\rangle$ where:

$$D^+ = \{X \subseteq W | D \subseteq X\}. \tag{5.9}$$

Theorem 5.2.16 (Zakharyaschev & Chagrov [1997, p.216]). *Every finite modal matrix is isomorphic to the dual of some finite frame with distinguished elements.*

Theorem 5.2.17. *Two non-normal modal logics are not translationally equivalent if they differ in the number of non-isomorphic frames w.d.e. of a given cardinality that validate them.*

Proof. If S and S' differ in the number of non-isomorphic frames with distinguished elements which validate them, then by Theorem 5.2.16 they will differ in the number of non-isomorphic modal matrices which validate them. But coalescent classes of matrices have the same number of matrices of a given cardinality, the function mapping matrices with a common definitional expansion being a bijection between the two classes preserving the cardinality of the underlying algebras. So the matrices for S and S' cannot be coalescent – and thus by Theorem 5.2.15 (contraposed) it follows that S and S' are not translationally equivalent. □

This result, coupled with the consideration of the three element frames with distinguished elements for S_i and S_j allow us to conclude the following.

Corollary 5.2.18. *The non-normal modal logics S_i and S_j are intertranslatable but not translationally equivalent.*

This example is not completely satisfying for a number of reasons. Firstly, it is dealing with non-normal modal logics and not normal modal logics – but this is a minor concern. Of more importance is that fact that it is not clear that there is a set of equivalence formula in both of the logics above of the form $\{C_1(p,q), C_2(p,q), \ldots, C_n(p,q)\}$ where each of the formulas C_i involve only some primitive binary connective of the two logics – no matter which set of boolean primitives we take – this being what Pelletier and Urquhart's requirement that logics share a common equivalence connective amounts to. While logics failing to be translationally equivalent due to not having an equivalence connective in their language isn't ruled out by our definitions, we could just as easily have stated our conditions so that this was so – ruling our this putative counterexample to the claim that intertranslatability is not translational equivalence. As it happens this is exactly how translational equivalence is stated in Pelletier & Urquhart [2003] – with S and S' being forced to be similarly equivalential.

One might perhaps think that this brings into question the adequacy of the definition of translational equivalence given by Pelletier and Urquhart, and as it happens things are much stranger than there merely being logics which fail to be the right sort of logics to be considered equivalent. Consider the logics **KD45** and **S4.4** with the connectives $\{\rightarrow, \neg, \Box\}$ primitive in case 1 and the connectives $\{\rightarrow, \neg, \leftrightarrow, \Box\}$ primitive in case 2. According to the official definition given in Pelletier & Urquhart [2003] **KD45** and **S4.4** cannot be translationally equivalent in case 1 as there is no primitive equivalence connective in the language, while they clearly can be in case 2. This kind of language variance affecting our judgments of equivalence is undesirable, and we will have something further to say about this in a later section. We can of course rule out this kind of variance by allowing our set of equivalence formulas Δ be any set of equivalence formulas which satisfy certain conditions (that $\tau(\Delta) = \Delta$ for example). Alternatively we could further free ourselves from language variance issues by simply requiring that the formulas A and $\tau'(\tau(A))$ be synonymous in the appropriate logic, dropping the requirement that their synonymy be marked by the provability of any particular formula(s) in the logic.

Taking this path we end up with a notion which is called 'definitional equivalence' in Wójcicki [1988]. There two logics S and S' are said to be *definitionally equivalent* whenever they share a common definitional extension. The idea of equivalent logics sharing a common definitional extension does quite a lot of work in Pelletier & Urquhart [2003] – where it is shown that logics are translationally equivalent iff they share a common definitional extension – being the notion at the heart of their translational invariants (as can be seen above).[13] The above considerations can be taken to bear upon the question, raised as Problem 1 in Pelletier [1984], of whether translational equivalence is a reasonable notion which captures the intended force of "really the same system". It is quite clear that the notion of translational equivalence is perfectly reasonable when considering congruential modal logics with the classical biconditional among their primitive connectives. Moreover, Pelletier and Urquhart make it quite clear that these logics (and others like them) are those with which they are concerned. So, at least in the context of normal modal logics it is definitely a reasonable notion, especially if we weaken our requirements on our set of equivalence formulas to appropriately avoid the kinds of language variance issues mentioned above. That said, we encounter some serious problems when we attempt to apply this notion to logics which do not have a connective among their stock of primitive connectives which licences replacement of formulas in all contexts. We will hold off for the moment on consideration as to whether this captures the intended force of "really the same system" until the next section. One thing we will say about this though, is that if this were the case (that translational equivalence properly captured the intended force of "really the same system") then the consequences would be, perhaps, somewhat odd. For example consider the weak modal logic **S0.5**, whose sole congruence is identity – which is definable as $\Box(\Box p \to \Box q)$[14]. We could either say that this logic

[13] Of course we have to be careful here regarding what counts as a definitional extension, and to go with their notion of translational equivalence Pelletier and Urquhart have a similar notion of a definition couched in terms of equivalence connectives rather than certain formulas being congruential.

[14] See Porte [1980] for more information.

fails to be a candidate to stand in the 'is translationally equivalent to' relation, or that it stands in that relation to no logic – making it, in a sense, truly unique. Not only that, but this situation would be quite common among the non-normal modal logics – all of them standing alone. While on the topic of Problem 1 in Pelletier [1984], the examples given in this chapter also allow us to show that translational equivalence is not trivial, if this is taken to mean that it is not a relation which holds either between all pairs of logics, or none of them. Not only are there pairs of logics which fail to be translationally equivalent (take S_i and S_j, **GL** and **Grz** or **KT** and **K** for example), but there are also pairs of logics which are translationally equivalent (**S4.4** and **KD45** for example). So we can see that translational equivalence is a non-trivial property of pairs of logics at the very least in this sense.

Returning to the topic at hand, the above example of S_i and S_j illustrates that, not only are these logics not translationally equivalent, but that they are not definitionally equivalent – failing to have a common definitional extension. As mentioned above though, these logics can also be considered to be invalid candidates for translational equivalence, and so it would be nice to have an example of a pair of monomodal logics which were similarly equivalential, intertranslatable and not translationally equivalent. This leaves us with the following open question.

Open Question 5.2.19. Is there a pair of normal monomodal logics which are similarly equivalential according to an equivalence connective which is primitive in each logic such that this pair of logics are intertranslatable but not translationally equivalent?

5.3 Translational Equivalence as Equivalence Between Logics

One philosophical motivation that is often put forward for thinking about translational equivalence is the thought that it captures our informal notion of what it is for two logics to be 'really the same logic'. Often when this is done, no justification is given as to why it is exactly this notion which captures our folk theory of equivalence between logics. For example, in Pelletier [1984] we are presented with the following quote.[15]

> "So $\mathbf{KT}^{\nabla}$ and $\mathbf{KT}$ are translationally equivalent; and, I would claim, this makes them the same logic." [Pelletier 1984, p.432].

No justification is given as to why this should be so. Moreover, whether translational equivalence properly captures our notion of two logics being "really the same", a notion which we will refer to henceforth as *pre-theoretic equivalence* between logics, is the first of the six problems which figure in the title of Pelletier [1984]. With this in mind, what we will do here is attempt to provide a systematic discussion of what formal notion best corresponds to pre-theoretic equivalence. To do this we will first begin by attempting to properly explicate our notion of pre-theoretic equivalence, using this discussion to discount various characterizations of what pre-theoretic equivalence could be.

The majority of the discussion in this section will be conducted at the level of logics as consequence relations, with some morals drawn for the logics as sets of formulas case a the end. Before continuing on, though, we should probably clarify what it is for two consequence relations to be translationally equivalent. Let us say that $\vdash_1$ and $\vdash_2$ are translationally equivalent whenever they are rendered intertranslatable by translations τ_1 and τ_2 for which the following holds for all formulas A, where $p \leftrightarrow q$ is an equivalence formula common to both $\vdash_1$ and $\vdash_2$.

[15] I have changed the labels for logics to match those used above – replacing Pelletier's V with $\mathbf{KT}^{\nabla}$ and T with $\mathbf{KT}$.

$$\vdash_1 A \leftrightarrow \tau_2(\tau_1(A)).$$
$$\vdash_2 A \leftrightarrow \tau_1(\tau_2(A)).$$

One first attempt which one might make at trying to formalize the notion of pre-theoretic equivalence is to follow the, admittedly somewhat naive, intuition that when two logics are the same then they have the same theorems, or contain the same sequents.

> Naive Proposal: Two logics $\vdash_1$ and $\vdash_2$ are pre-theoretically equivalent whenever they contain the same sequents.

In order to assess how reasonable, or not, this proposal is we need to be clear about what it means for two logics, in possibly different languages, to validate the same sequents. We could take the strict approach and say that two sequents are the same whenever they contain the same symbols arranged in the same order – but this would mean that trivial changes in notation would have consequences relating to the sameness of logics, which seems remarkably implausible. A second attempt we could make would be something as follows. Let us say that two logics $\vdash_1$ and $\vdash_2$ validate the same sequents whenever there if a function f which maps each primitive n-ary connective of $\vdash_1$ to an n-ary connective of $\vdash_2$ such that $\Gamma \vdash_1 B$ iff $f(\Gamma) \vdash_2 f(B)$.

To see that the Naive Proposal, as spelled out above, is incorrect consider the following example. Given a set of valuations V let us say that $\vdash_V$ is the unique consequence relation such that $A_1, \ldots, A_n \vdash_V B$ iff for all $v \in V$ if $v(A_1) = T$ and ...and $v(A_n) = T$ then $v(B) = T$. Let V_1 be the set of all valuations on the propositional language built up from $\{\vee, \neg\}$ which satisfy conditions $(v_\vee)$ and $(v_\neg)$, and V_2 be the set of all valuations on the propositional language built up from $\{\rightarrow, \neg\}$ which satisfy conditions $(v_\rightarrow)$

and $(v_\neg)$.

$$(v_\neg) \quad v(\neg A) = T \quad \text{if and only if} \quad v(A) = F.$$
$$(v_\vee) \quad v(A \vee B) = T \quad \text{if and only if} \quad v(A) = T \text{ or } v(B) = T.$$
$$(v_\rightarrow) \quad v(A \rightarrow B) = F \quad \text{if and only if} \quad v(A) = T \text{ and } v(B) = F.$$

It is easy to see that $\vdash_{V_1}$ and $\vdash_{V_2}$ are pre-theoretically equivalent – these are just the $\{\vee, \neg\}$ and $\{\rightarrow, \neg\}$-fragments of classical propositional logic. However, these two logics do not verify the same sequents in the sense described above, as $\vdash_{V_2} p \rightarrow (q \rightarrow p)$, as there is only one binary connective in each language $f(A \rightarrow B) = f(A) \vee f(B)$, but $\nvdash_{V_1} p \vee (q \vee p)$. So clearly the Naive proposal will not do.

The main intuition which was driving our conclusion that $\vdash_{V_1}$ and $\vdash_{V_2}$ are pre-theoretically equivalent was that we could define the connectives of one in terms of those of the other, and vice versa. That is to say, the intuition which was behind this judgment was something like the following.

> Intertranslatability Proposal: Two logics $\vdash_1$ and $\vdash_2$ are pre-theoretically equivalent whenever there are definitional translations τ_1 and τ_2 which render them intertranslatable.

The reason why we are concerned with definitional translations here is because of their relationship to definitions – definitional translations between logics allow us to see how to define the connectives of the source logic in the language of the target logic. It is easy to see that there are definitional translations which render $\vdash_{V_1}$ and $\vdash_{V_2}$ intertranslatable, so we currently have no conflicts with our accepted data. At this point, though, there is a potential objection one could level against the Intertranslatability Proposal – namely that this proposed characterisation is too strong – and it is this objection which we will address now.

5.3.1 Makinson's Warning

One could readily agree that what is needed for two logics to be pre-theoretically equivalent was for them to be intertranslatable, but object to the Intertranslatability Proposal as it is outlined above on the grounds that this characterisation is, nonetheless, too strong. The objection here is that we don't necessarily need our translations to be definitional ones, surely any variable-fixed translation would do. This gives rise to the following counter-proposal.

> Weak-Intertranslatability Proposal: Two logics $\vdash_1$ and $\vdash_2$ are pre-theoretically equivalent whenever there are variable-fixed translations τ_1 and τ_2 which render them intertranslatable.

This proposal is usually accompanied by a certain view of what constitutes a definition of a connective, and can be seen being advocated most clearly in Segerberg [1982].[16] The motivation for this view, over the Intertranslatability Proposal, could be seen to stem from the results appearing in Makinson [1973]. What we will show here is that, in actual fact, the anomaly which was noticed by Makinson actually shows us why we should reject the Weak-Intertranslatability Proposal.

Let $L1$ be the propositional language built up in the usual way from a countable supply of propositional variables using the connectives $\{\to, \bot, \Box\}$, and $L2$ the propositional language built using the connectives $\{\to, \neg, \Box\}$. Define V as the set of all valuations in the language $L1$ which satisfy condition $(v_\to)$ in addition to the condition $(v_\bot)$, and V' as the set of all valuation in the language $L2$ which satisfy conditions $(v_\to)$ and $(v_\neg)$.

$$(v_\bot) \quad v(\bot) = F$$

[16] Even more radically, Segerberg's proposal allows for arbitrary translations. As it happens though, our objection to the Weak-Intertranslatability Proposal will carry over to the view which merely requires logics to be intertranslatable with no restrictions on the translations.

Following the treatment of Makinson [1973] given in Segerberg [1982] we have the following.[17]

Theorem 5.3.1. *$\vdash_V$ and $\vdash_{V'}$ are rendered intertranslatable by variable-fixed translations.*

Proof. The above result is easy to show, the translations in question being as follows where τ_V and $\tau_{V'}$ are variable-fixed and homonymous on $\to$ and $\Box$.

$$\tau_V(\bot) = \neg(p_0 \to p_0). \qquad \tau_{V'}(\neg A) = \tau_{V'}(A) \to \bot.$$

$\Box$

Theorem 5.3.2 (Segerberg [1982, p.102]). *$\vdash_V$ is an intersection of two of its proper extensions.*

In particular we can show that $\Gamma \vdash_V B$ iff $\Box\bot, \Gamma \vdash_V B$ and $\Box\bot \to \bot, \Gamma \vdash_V B$.

Theorem 5.3.3 (Segerberg [1982, p.103]). *$\vdash_{V'}$ is not the intersection of any two of its proper extensions.*

Krister Segerberg has the following to say about this situation.

> "There are several ways to react to Makinson's Warning. One is to go on finding it disturbing and accept, as a fact of life, that logic is language-sensitive in the way we have just seen." Segerberg [1982, p.104].

[17]Technically Segerberg appears to be thinking of equivalence between logics as requiring a condition similar to equipollence below, but with no restriction placed upon the structure of translations. Segerberg's account of syntactic equivalence will coincide with equipollence (considered below) when we require that the translations in question are compositional, and with definitional equivalence if we require that the translations are compositional and the the underlying logic is congruential.

As is alluded to in the above passage, our notion of pre-theoretic equivalence is one which is not so language-sensitive, and it would be disturbing indeed if, in the general case, we had to accept as much. For this to really be so disturbing two things are required though, firstly the results above – which are clearly fine – and secondly the idea that Weak-Intertranslatability Proposal is correct – and that logics which are intertranslatable by variable-fixed translations are pre-theoretically equivalent. But why should we think this at all? Surely the above results simply show that this cannot be the case, as logics which are intertranslatable in this way can still vary quite wildly in ways which make this look implausible as an explication of pre-theoretic equivalence. So it appears that 'Makinson's Warning' is a warning indeed, but not regarding the language-sensitivity of logic.[18]

Thus far we have argued that our notion of pre-theoretic equivalence is at least as strong as that given by the intertranslatability proposal. The question then remains as to whether this properly captures our notion of sameness between logics – after all it gives us an equivalence relation (Theorem 5.0.9) which agrees with our evidence concerning some candidate logics which we think are pre-theoretically equivalent. There are two related treatments present in the literature which extend the Intertranslatability proposal, both of which are motivated by similar ideas about pre-theoretic equivalence. It is these two views which we will examine now.

What sort of properties can a logic have? A host of different properties are mentioned in the literature – closure under this or that rule, the interpolation property, being pre-tabular etc. Does our theory of what it is for two logics to be pre-theoretically equivalent force them to share all the same properties? That is to say, are some properties of logics properties of a particular representation or presentation of a logic, rather than being

[18]This line of argument is influenced extensively by §3.3 of Humberstone [1993], the point at issue there having to do with definitions rather than equivalence between logics.

tied up intimately with the logic itself? This is not entirely clear, but that aside, we might reason that as pre-theoretic logics are 'really just the same logic' in some sense, then we should be able to point at the logic which they are both just syntactic variants of. That is to say they should share a definitional extension, giving us the following Condition.

> Extension Condition: $\vdash_1$ and $\vdash_2$ are pre-theoretically equivalent whenever they share a common definitional extension.

If we take the pre-theoretically equivalent logics to be those which are intertranslatable which also satisfy the extension condition then a picture of what pre-theoretic equivalence between logics is begins to emerge. Say that two consequence relations $\vdash_1$ and $\vdash_2$ are *Definitionally Equivalent* whenever they are rendered intertranslatable by translations τ and τ' such that for all formulas A, A and $\tau'(\tau(A))$ are synonymous in $\vdash_1$ and A and $\tau(\tau'(A))$ are synonymous in $\vdash_2$ – where in this context two formula A and B are synonymous according to $\vdash$ whenever (with the $C(\cdot)$ notion understood as wiht the statement above) $C_1(A),\ldots,C_n(A) \vdash C_{n+1}(A)$ iff $C_1(B),\ldots,C_n(B) \vdash C_{n+1}(B)$.

Theorem 5.3.4 (Wójcicki [1988]). *(i) If* S *and* S' *are intertranslatable logics which share a common definitional extension, then* S *and* S' *are definitionally equivalent. (ii) If* $\vdash_1$ *and* $\vdash_2$ *are intertranslatable logics which share a common definitional extension, then* $\vdash_1$ *and* $\vdash_2$ *are definitionally equivalent.*

One corollary of this result allows us to settle Problem 1 from Pelletier [1984].

Corollary 5.3.5. *If* S *and* S' *are intertranslatable normal modal logics which share a common definitional extension then* S *and* S' *are translationally equivalent.*

Our rival extension of the Intertranslatability Proposal also arises from an intuition concerning what we have when two logics are pre-theoretically equivalent. In this case the motivation has to do with equivalent logics having isomorphic theory spaces – each theory of the one logic being able to be mapped to a theory of the other in a way which preserves their interrelationships. To be more precise, given a consequence relation $\vdash$ and a set of formulas Γ in the language of $\vdash$ let the set $\Gamma^{\vdash} = \{A | \Gamma \vdash A\}$. Call a set of formulas Δ a $\vdash$-*theory* whenever $\Delta^{\vdash} = \Delta$, letting $Th(\vdash)$ denote the set of all $\vdash$-theories. Then the alternative proposal for what makes logics equivalent, called 'equipollence' in Caleiro & Gonçalves [2007] is the following.

> Theory-Extension Condition: Two consequence relations $\vdash_1$ and $\vdash_2$ are equivalent whenever they are intertranslatable and $Th(\vdash_1)$ and $Th(\vdash_2)$ are isomorphic.

Following Caleiro & Gonçalves [2007] we will call pairs of logics, considered as consequence relations which satisfy the Theory-Extension condition 'equipollent', and note that we can characterize equipollence in the following way.[19]

Theorem 5.3.6. *Suppose that $\vdash_1$ and $\vdash_2$ are consequence relations. Then $\vdash_1$ and $\vdash_2$ are equipollent iff they are rendered intertranslatable by translations τ_1 and τ_2, and additionally fulfil the following conditions.*

$$\begin{aligned} A &\dashv\vdash_1 \tau_2(\tau_1(A)). \\ A &\dashv\vdash_2 \tau_1(\tau_2(A)). \end{aligned}$$

Now it is clear that this notion is weaker than definitional equivalence, as every pair of definitionally equivalent consequence relations are also

[19] The notion under discussion in Kuhn [1977] – which is essentially what we are calling equipolence above – will only coincide with translational equivalence when the logics involved are assumed to be congruential, and also have an implication connective in their object language which satisfied both modus ponens and the deduction theorem. We presume that these kinds of conditions are what Pelletier & Urquhart [2003] have in mind when they claim that these two notions are equivalence.

equipollent.[20] Whether equipollence is strictly weaker than definitional equivalence is an open question.

One thing which is indeed interesting and bears on the above question, though, is that these two notions collapse whenever the consequence relations involved are *congruential* in the sense that A and B are synonymous according to $\vdash$ exactly when $A \dashv\vdash B$.

Theorem 5.3.7. *Suppose that $\vdash_1$ and $\vdash_2$ are congruential. Then $\vdash_1$ are $\vdash_2$ are definitionally equivalent iff they are equipollent.*

So clearly if equipollence is strictly weaker than definitional equivalence the logics involved in showing this would have to be non-congruential. So we can see that whatever pre-theoretic equivalence is, it lies somewhere in between equipollence and definitional equivalence. While this may seem rather weak it does give us some tools for determining whether some logics are or are not pre-theoretically equivalent. For example we know that, if two logics are definitionally equivalent then they will be pre-theoretically equivalent, and that if they fail to be equipollent then they will not be pre-theoretically equivalent. This allows us to say with some certainty, for example, that classical propositional logic and intuitionistic propositional logic are indeed different logics – as these two logics fail to be intertranslatable (and hence fail to be equipollent), as noted in Theorem 2.6.9 of Wójcicki [1988].

Where, then, does this leave the sentiment expressed in quote from Pelletier [1984] with which we opened this section. Since logics which are translationally equivalent are definitionally equivalent it is certainly true. Although, again, the problems of determining what pre-theoretic equivalence is remain even at the propositional level – indeed they are ex-

[20]Proof: Suppose that $\vdash_1$ and $\vdash_2$ are definitionally equivalent – and that τ_1 and τ_2 are the translations according to which they are so. Then we know that A and $\tau_2(\tau_1(A))$ are synonymous according to $\vdash_1$ and that $A \dashv\vdash_1 A$ by reflexivity, and so by synonymity we have that $A \dashv\vdash_1 \tau_2(\tau_1(A))$. A similar argument gives us the corresponding condition for $\vdash_2$.

acerbated by difficulties with determining what the equivalent notion to equipollence is in the FMLA framework. This does put us in a situation to address Problem 1 from Pelletier [1984], as to whether translational equivalence captures the notion of 'really the same logic'. The answer really comes down to how we're thinking about logic here – because two logics could be definitionally equivalent, and hence be pre-theoretically equivalent, while failing to have a set of equivalence formulas definable in them – let alone a common equivalence connective. We can see an example of this kind of phenomenon among consequence relations in Caleiro & Gonçalves [2007], where it is shown that the SET-FMLA logics of classical disjunction and classical ternary disjunction are equipollent, but not translationally equivalent. As it happens, though, these two logics are definitionally equivalent, so we can imagine a similar situation occurring in propositional logic in the FMLA framework. So translational equivalence isn't pre-theoretic equivalence, but we can remain confident that whenever two logics are translationally equivalent they are pre-theoretically equivalent. That is, translationally equivalent logics 'really are the same logic'.

VI

Translations in Non-Normal Modal Logics

In this chapter we will be concerned with issues arising when we allow our source and/or target logics to be non-normal modal logics. There are a number of reasons why we might find such translations of interest, some formal and some philosophical. For example, the original Lewis modal systems **S2** and **S3** are both non-normal systems whose semantics, involving non-normal worlds, are somewhat cumbersome (Lemmon [1966]). One fact which those semantics do make obvious, though, is the relationship between **S2** and **KT**, and **S3** and **S4**. This relationship was first noted in Aanderaa [1969], where the following two translations are considered, both of which are homonymous on the classical connectives (but not, as you will note, variable-fixed).

$$\tau_-(p_i) = p_{i+1} \qquad \tau_-(\Box A) = \Box\tau_-(A) \wedge p_0.$$
$$\tau_+(p_i) = p_i \qquad \tau_+(\Box A) = \Box\top \to \Box\tau_+(A).$$

Before we continue on, it is worth noting that already τ_- is not of the kind we have been primarily concerned with – those being definitional translations, which are both compositional and also variable-fixed.[1] It deviates further from the sorts of translations we have been considering though, in that what Aandrea showed was not that $A \in \mathbf{S2}$ if and only if $\tau_-(A) \in \mathbf{KT}$, but rather the following.

Theorem 6.0.8. *For all formulas A, we have the following.*

(ai)	$A \in \mathbf{S2}$	*if and only if*	$p_0 \to \tau_-(A) \in \mathbf{KT}$.
(aii)	$A \in \mathbf{KT}$	*if and only if*	$\tau_+(A) \in \mathbf{S2}$.
(bi)	$A \in \mathbf{S3}$	*if and only if*	$p_0 \to \tau_-(A) \in \mathbf{S4}$.
(bii)	$A \in \mathbf{S4}$	*if and only if*	$\tau_+(A) \in \mathbf{S3}$.

As mentioned in Chapter 2, translations like τ_- are technically T_5 type translations, despite being obviously different from most translations which we might be concerned with. Largely, though, we will continue to focus on definitional translations even among the realms of non-normal modal logics.

One area in philosophical logic where people are often concerned with definitional translations involving non-normal modal logics is in the study of contingency operators. The usual problem here is to determine what the logic of the operator $\nabla p =_{def} \Diamond p \wedge \Diamond\neg p$ – interpreted to mean 'it is contingent whether p' – is over a given normal modal logic S (i.e. what $\mathsf{S}(\nabla)$ is). It is easy to show that for all consistent normal modal logics $\mathsf{S}(\nabla)$ will not be normal. To see this consider the τ_∇-translation of $\Box\top$. This will be the formula $\Diamond\top \wedge \Diamond\bot$. The first conjunct will be provable in all extensions of **KD**, but the second conjunct is not provable in any consistent normal modal logic. This means that the problem of determining the contingency fragment of a given normal modal logic is the problem of finding the unique non-normal modal logic S' such that, for all formulas A we

[1] We will meet τ_+ again soon, where we will more suggestively call it $\tau_{\boxplus}$.

have the following.

$$A \in \mathsf{S}' \text{ if and only if } \tau_{\nabla}(A) \in \mathsf{S}.$$

For a wide range of common normal modal logics this question has been answered (Kuhn [1995] and Humberstone [1995]). One might also wonder which logics S can be faithfully embedded into their own contingency fragment. This question is considered at length in Cresswell [1988], where it is shown that all normal extensions of **KT** can be faithfully embedded into their own contingency fragments via the translation $\tau_{\boxdot}$. Cresswell also gives an example of a normal modal logic which can be faithfully embedded into its own contingency fragment which isn't an extension of **KT** – namely the normal extension of **K** by the following formula.

$$\mathbf{Cr}: \quad \Box p \leftrightarrow ((\Box p \vee \Box\neg p) \wedge (p \leftrightarrow (\Box(\Box p \vee \Box\neg p) \vee \Box\neg(\Box p \vee \Box\neg p))))$$

Using the translation which translate $\Box p$ as $\neg\nabla p \wedge (p \leftrightarrow \neg\nabla\neg\nabla p)$[2] we are able to faithfully embed **KCr** into **KCr**(∇). That **T** $\notin$ **KCr** is easy to show, the interested reader being referred to Cresswell [1988].

The rest of this chapter is broken up into three sections. First in §1 we will look at some issues which arise concerning the modal logic of (morally relevant) ability – which Anthony Kenny persuasively argued in Kenny [1976] must be non-normal. In particular we will focus on some issues which arise out of a paper by M. A. Brown. In §2 we will look at a translation due to S.K. Thomason which faithfully embeds multi-modal logics into monomodal logics, focusing in particular on the applications of this translation to the problem of faithfully embedding **E** into **K**. We will close that section by looking at some other embeddings involving non-normal modal logics, before then going on in §3 to give a simplified version of the translations from §2.

[2]To see why this is the case note that we could have equivalently written **Cr** as the formula $\Box p \leftrightarrow (\neg\nabla p \wedge (p \leftrightarrow \neg\nabla\neg\nabla p))$, in this case taking ∇p as an abbreviation for $\Diamond p \wedge \Diamond\neg p$.

6.1 Translations and the Logic of Ability

We will begin our look at translations involving non-normal modal logics by considering a motivating example drawn from Brown [1988]. Therein an attempt is made to try and determine what the modal logic of (morally relevant) ability is – the modal logic for which we can read "$\diamondplus A$" as saying that I can bring about the circumstances in which A is true.[3] Brown, following up on comments made in Kenny [1976] notes that this logic cannot be an extension of the smallest normal modal logic **K**. To see this consider the following instance of **K**.

$$\mathbf{K}: \quad \diamondplus(p \vee q) \to (\diamondplus p \vee \diamondplus q).$$

Under the above interpretation **K** can be read as saying that if I can bring it about that $p \vee q$ is true, then I can bring about the truth either of p or of q. To see that this makes **K** false under the intended interpretation Kenny notes that, if presented with a deck of cards I am able to bring it about that the card I pick will be red or the card I pick will be black is true. I am not, on the other hand (assuming that we are dealing with a standard deck of cards) able to bring it about that the card I pick will be red is true, nor can I bring it about that the card I pick will be black is true.

With these considerations in mind Brown considers the possibility of treating the logic of the **can** of ability as being an extension of **E** the smallest congruential modal logic.[4] Brown introduces the following semantic clause for the modal operator which we have written as $\diamondplus$.

$$\mathcal{M} \models_x \diamondplus A \iff \exists X \in N(x) \text{ such that } \forall y \in X\ \mathcal{M} \models_y A. \tag{6.1}$$

[3]We could equally well have introduced our intended interpretation by relativizing our modal operator to an agent, as is often done in epistemic logic. Then we would be saying that "$\diamondplus_a A$" means that agent a can bring about the circumstances in which A is true. We will leave our modal operators unsubscripted here in the interests of brevity.

[4]The following material appears in French [2009]

Here we are thinking of our semantic structures as being neighbourhood models. We will call a structure $\mathcal{M} = \langle W, N, V \rangle$ a *neighbourhood model* whenever W is a non-empty set, N a function from W to $\wp(\wp(W))$, and V a function from the propositional variables to subsets of W.[5] For convenience we will often refer to sets $X \in N(x)$ as the *neighbourhoods of* x. The idea behind the above truth clause is that we are to think of the neighbourhoods of x as being the "outcomes" of our unspecified agent's possible actions in the situation x. An agent is thus able to bring it about that A if there is an outcome relative to x throughout which A is true.

Brown's full logic also contains an operator $\diamondtimes$ which can be thought of as expressing **might** – where $\boxplus A$ expresses that I **will** do A. The semantic clause Brown gives for his **might** operator $\diamondtimes$ is the following.

$$\mathcal{M} \models_x \diamondtimes A \iff \exists X \in N(x) \text{ such that } \exists y \in X \text{ and } \mathcal{M} \models_y A. \qquad (6.2)$$

We can characterize Brown's logic of action and ability, called $\mathcal{V}$ in honour of Peter van Inwagen, as being the set of all formulas A constructed out of the propositional language containing the operators $\diamondplus$ and $\diamondtimes$ which are valid on all neighbourhood frames when we evaluate $\diamondplus$ and $\diamondtimes$ according to (6.1) and (6.2).

Having introduced the above semantic clause (6.1) for the modal operator $\diamondplus$ Brown then goes on to say that:

> "We could also give a complete axiomatization of a subsystem using only the operator $\diamondplus$ and its dual. The obvious adaptation of the classical system **E** would serve." [Brown 1988, p.14].

The "subsystem using only the operator $\diamondplus$" (which we will henceforth call $\mathcal{V}_{\diamondplus}$) would of course be the set of all formulas A containing only the operator $\diamondplus$ (in addition to the classical connectives) which are valid on all neighbourhood frames. It is clear from the context in which Brown says

[5] The neighbourhood semantics appear in Chellas [1980, p.207f.], where neighbourhood models are referred to as minimal models.

this that he is taking the 'obvious adaptation' to be that of replacing all occurrences of $\oplus$ and its dual $\boxtimes$ with $\Diamond$ and $\Box$. This claim is simply untrue – it is not the 'obvious adaptation' of **E**, but rather that of **EM** the smallest monotonic modal logic which would serve in this regard. Before going on to show that this is the case it is worth considering what the diagnosis for Brown's mistake is. Consider the following standard semantic clause for $\Box$ in a neighbourhood model $\mathcal{M}$ – where $\|A\|$ is the set of all points $x \in W$ such that A is true at x in $\mathcal{M}$.[6]

$$\mathcal{M} \models_x \Box A \iff \|A\| \in N(x). \tag{6.3}$$

It is well known that the logic determined by the class of all neighbourhood frames when $\Box$ is interpreted according to (6.3) is the smallest congruential modal logic **E**. By contrast, in the above quotation Brown is saying that the modal logic determined by the class of all neighbourhood frames interpreted according to (6.1) is also **E**, when this in fact picks out **EM**.

A similar semantics for weak modal logics to that considered by Brown is given in Jennings & Schotch [1981]. There the semantic structures under consideration are locale frames, where $\langle W, N\rangle$ is a *locale frame* if $W \neq \emptyset$ and $N : W \to \wp(\wp(W))$, where N also satisfies the following condition (called *Minimality*).

$$\forall w \in W, \forall X \subseteq W(X \in N(w) \Rightarrow \forall Y(Y \subseteq X \Rightarrow Y \notin N(w))).$$

In Jennings & Schotch [1981] only logics which are determined by locale frames which also satisfy the condition that $N(w) \neq \emptyset$ for all $w \in W$ (which corresponds to the validity of the formula **N** $(=\Box\top)$) are under consideration, with the evaluation clause for $\Box B$ being as follows, where where

[6]Whenever there is any ambiguity as to which model we are referring to when we say $\|A\|$ we will use a superscript to indicate the model in question, making the above definition be for $\|A\|^{\mathcal{M}}$

$\mathcal{M} = \langle W, N, V \rangle$ is a model on a locale frame.

$$\mathcal{M} \models_x \Box B \text{ if and only if } \exists X \in N(x) \text{ such that } X \subseteq \|B\|. \tag{6.4}$$

Schotch & Jennings show that the logic determined by the class of all such locale frames when we evaluate $\Box B$ according to (6.4) is the modal logic **EMN**. As it is the condition that $N(w)$ be non-empty for all $w \in W$ which makes **N** valid, this would suggest that the smallest modal logic determined by the class of all locale frames would be exactly **EM**. Given Theorem 6.1.2 this would lead us to believe that the $[Minimality]$ condition above is doing no real work insofar as determining what formulas are valid on locale frames – at least when we place no further conditions upon L.

Leaving this comparative line of reasoning aside we will now go on to show that the set of all formulas valid on the class of all neighbourhood frames using the evaluation clause (6.1) (with $\oplus$ reconstrued as $\Box$) is the weakest monotonic modal logic **EM** – the congruential extension of the logic **E** by the following formula **M**.

$$\mathbf{M}: \quad \Box(A \wedge B) \rightarrow (\Box A \wedge \Box B).$$

Let us call $N(x)$ the *neighbourhood set of x*, and say that a neighbourhood set is *supplemented* whenever it fulfils the following condition:

$$\text{if } X \subseteq Y \text{ and } X \in N(x) \text{ then } Y \in N(x).$$

It is well known that the logic **EM** is determined by the class of all neighbourhood frames whose neighbourhoods are supplemented (henceforth, *supplemented neighbourhood frames*), this result appearing as Theorem 9.10 of Chellas [1980, p.258].

Definition 6.1.1 (Chellas [1980]). Given a neighbourhood model $\mathcal{M} = \langle W, N, V \rangle$ let the *supplementation of* $\mathcal{M}$ be the model $\mathcal{M}^+ = \langle W, N^+, V \rangle$, where N^+ is the closure of N under supersets. That is to say, $X \in N^+(x)$ iff $Y \subseteq X$ for some $Y \in N(x)$.

Theorem 6.1.2. *Let $\mathcal{M}$ be a neighbourhood model, and $\mathcal{M}^+$ be its supplementation. Then for all formulas A and all points $x \in W$ we have the following:*

$$\mathcal{M} \models_x \diamondplus A \text{ if and only if } \mathcal{M}^+ \models_x \Box A.$$

Where $\diamondplus$ is evaluated according to (6.1) and $\Box$ according to (6.3).

Proof. For the 'only if' direction suppose that $\mathcal{M} \models_x \diamondplus A$. Then there is a set $X \subseteq \|A\|$ such that $X \in N(x)$. So as $N(x) \subseteq N^+(x)$ then $X \in N^+(x)$. As $\|A\| \supseteq X$ we know that $\|A\| \in N(x)$ it follows that $\mathcal{M}^+ \models_x \Box A$.

For the 'if' direction suppose that $\mathcal{M}^+ \models_x \Box A$. So we know that $\|A\| \in N^+(x)$ so by the definition of $N^+(x)$ we know that there is an $X \subseteq \|A\|$ such that $X \in N(x)$ and thus that $\mathcal{M} \models_x \diamondplus A$. □

What we will now show is that the logic of the modality $\Diamond$ over $\mathcal{V}_{\diamondplus}$ is the smallest monotonic modal logic **EM** (Theorem 6.1.3). We will then go on to show that $\Diamond$ and $\Box$ are analogous over **EM** (Theorem 6.1.5), and thus that $\mathcal{V}_{\diamondplus} = \mathbf{EM}$ (Theorem 6.1.6).

Theorem 6.1.3. $\mathcal{V}_{\diamondplus}(\Diamond) = \mathbf{EM}$.

Proof. Follows from Theorem 6.1.2 and the fact that **EM** is determined by the class of all supplemented neighbourhood frames. □

Lemma 6.1.4. *Let $\mathcal{N} = \langle W, N, V \rangle$ be a supplemented neighbourhood model, and $\mathcal{N}^d = \langle W, N^d, V \rangle$ be the model in which, for all $X \subseteq W$ and $x \in W$ we have that $X \in N(x) \iff W \smallsetminus X \notin N^d(x)$. Then for all formulas A and points $x \in W$ we have the following:*

$$\mathcal{N} \models_x A \text{ if and only if } \mathcal{N}^d \models_x \tau_{\Diamond}(A)$$

Proof. By induction upon the complexity of A – the only case of interest being that where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that $\mathcal{N} \models_x \Box B$. Then we know that $\|B\| \in N(x)$. By the inductive hypothesis we know that for all such points

$z \in \|B\|$ that $\mathcal{N}^d \models_x \tau_\diamond(B)$. By the definition of N^d we know that $W \smallsetminus \|B\|^{\mathcal{N}} \notin N^d(x)$, and that as $\|B\|^{\mathcal{N}} = \|\tau_\diamond(B)\|^{\mathcal{N}^d}$ we know that $W \smallsetminus \|\tau_\diamond(B)\|^{\mathcal{N}^d} \in N^d(x)$, from which it follows that $\mathcal{N}^d \models_x \Diamond\tau_\diamond(B)$.

For the 'if' direction suppose that $\mathcal{N}^d \models_x \Diamond\tau_\diamond(B)$. It follows that $W \smallsetminus \|\tau_\diamond(B)\| \notin N^d$. By the inductive hypothesis and the definition of N^d it follows that $\|B\| \in N(x)$ and thus that $\mathcal{N} \models_x \Box B$.

□

Theorem 6.1.5. $\mathbf{EM}(\Diamond) = \mathbf{EM}$.

Proof. What we want to show is that, for all formulas A we have the following.

$$A \in \mathbf{EM} \text{ if and only if } \tau_\diamond(A) \in \mathbf{EM}. \tag{6.5}$$

The 'only if' direction follows from Definition 8.7 in Chellas [1980, p.234] and the fact that **EM** is closed under the rule **RM**$\diamond$ (Theorem 8.12(1) in Chellas [1980, p.238]).

For the 'if' direction suppose that $A \notin \mathbf{EM}$. Then there is a supplemented neighbourhood model $\mathcal{N}$ and a point $x \in W$ such that $\mathcal{N} \not\models_x A$. By Lemma 6.1.4 it follows that $\mathcal{N}^d \not\models_x \tau_\diamond(A)$, and that as $\mathcal{N}^d$ can be easily seen to be supplemented whenever $\mathcal{N}$ is it follows that $\tau_\diamond(A) \notin \mathbf{EM}$ and thus that $A \notin \mathbf{EM}(\Diamond)$ as desired. □

Theorem 6.1.6. $\mathcal{V}_{\oplus} = \mathbf{EM}$.

Proof. $\mathcal{V}_{\oplus}(\Diamond) = \mathbf{EM}$ by Theorem 6.1.3 , which by Theorem 6.1.5 means that $(\mathcal{V}_{\oplus}(\Diamond))(\Diamond) = \mathbf{EM}(\Diamond)$. It is easy to see that $(\mathsf{S}(\Diamond))(\Diamond) = \mathsf{S}$ for any modal logic S, and hence that $\mathcal{V}_{\oplus} = \mathbf{EM}$. □

6.1.1 Relationship between K and EM

One of the other results present in Brown [1988] is that we can faithfully embed $\mathcal{V}$ into **K** using a translation τ which interprets $\otimes$ as $\Diamond\Diamond$ and $\oplus$ as $\Diamond\Box$. We can consider τ as consisting of two separate translations, the source logics for which are the $\mathcal{V}_{\diamondtimes}$ and $\mathcal{V}_{\diamondplus}$ subsystems of the full system $\mathcal{V}$. Considering the two translations separately in this way provides us with some interesting results. Firstly the fact that **K** is iterative (i.e. that $\mathbf{K}(\Box^n) = \mathbf{K}$ for all $n \in Nat$, as shown on p.42) indicates to us in a quite direct way that the $\diamondplus$-subsystem of $\mathcal{V}$ is just the logic **K**. What is more interesting though is that the translation τ faithfully embeds the $\diamondplus$-subsystem of $\mathcal{V}$ in **K**. In light of the above result concerning the fact that the $\diamondplus$-subsystem of $\mathcal{V}$ is none other than **EM** we can recast this result as showing that $\tau_{\Diamond\Box}$ faithfully embeds **EM** into **K**.

What makes this result doubly interesting is that it is well known that we can faithfully embed **EM** into bimodal **K** (the modal logic with two modal operators $\Box_1$ and $\Box_2$ which are both modal operators for **K**) via the translation which replaces $\Box$ with $\Diamond_1\Box_2$ (Kracht & Wolter [1999]). We will return to this point later. In the interests of completeness we will give the proofs alluded to in Brown [1988, p.15f.], recasting them in terms of supplemented neighbourhood models using evaluation clause (6.3), rather than arbitrary neighbourhood models using evaluation clause (6.1).

Definition 6.1.7. Given a model $\mathcal{M} = \langle W, R, V\rangle$ where $\langle W, R\rangle$ is a relational frame, construct a neighbourhood model $\mathcal{N}_R = \langle W, N_R, V\rangle$ where:

$$N_R(x) = \{R(y) | y \in R(x)\}.$$

Theorem 6.1.8. *Let $\mathcal{M} = \langle W, R, V\rangle$ be a model on a relational frame $\langle W, R\rangle$, and $\mathcal{N}_R^+ = \langle W, N_R^+, V\rangle$ be the supplementation of the model $\mathcal{N}_R$. Then for all formulas A and all points $x \in W$ we have the following.*

$$\mathcal{M} \models_x \tau_{\Diamond\Box}(A) \textit{ if and only if } \mathcal{N}_R^+ \models_x A.$$

Proof. By induction upon the complexity of A, the only case of interest being that where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that $\mathcal{M} \models_x \Diamond\Box\tau_{\Diamond\Box}(B)$. The there is a point y such that Rxy and $\mathcal{M} \models_y \Box\tau_{\Diamond\Box}(B)$. Thus for all points z such that Ryz we know that $\mathcal{M} \models_z \tau_{\Diamond\Box}(B)$. By the inductive hypothesis we can reason that for all such points z, $\mathcal{N}_R^+ \models_z B$. As $R(y) \in N_R(x)$ and $R(y) \subseteq \|A\|$ it is clear that $\mathcal{N}_R^+ \models_x \Box B$.

For the 'if' direction suppose that $\mathcal{N}_R^+ \models_x \Box B$. Then we know that $\|B\| \in N_R^+(x)$. As all the neighbourhoods of N_R^+ are of the form $R(y)$ for some y such that Rxy it is clear that, for all such points $z \in R(y)$ we have that $\mathcal{N}_R^+ \models_z B$. By the inductive hypothesis we know that $\mathcal{M} \models_z \tau_{\Diamond\Box}(B)$. As these points are all in $R(y)$ we know that $\mathcal{M} \models_y \Box\tau_{\Diamond\Box}(B)$. Lastly, as Rxy it follows that $\mathcal{M} \models_x \Diamond\Box\tau_{\Diamond\Box}(B)$. □

For the following Theorem we will need the appropriate modification of the following elegant model construction given in Brown [1988, p.15], where we add new points corresponding to each neighbourhood which bear the accessibility relation to all of their members.

Definition 6.1.9. Given a neighbourhood model $\mathcal{N} = \langle W, N, V\rangle$ construct a model on a relational frame $\mathcal{M}_N = \langle W_N, R_N, V\rangle$ as follows.

- $W_N = W \cup \{\langle x, X\rangle | X \in N(x)\}$
- $R_N xy \iff (\exists X \subseteq W)[(y = \langle x, X\rangle) \vee \exists z \in W(x = \langle z, X\rangle \;\&\; y \in X)]$

Theorem 6.1.10. *Let $\mathcal{N} = \langle W, N, V\rangle$ be a supplemented neighbourhood model, and $\mathcal{M}_N = \langle W_N, R_N, V\rangle$ be as above. Then for all formulas A and all points $x \in W$ we have the following.*

$$\mathcal{N} \models_x A \textit{ if and only if } \mathcal{M}_N \models_x \tau_{\Diamond\Box}(A).$$

Proof. By induction upon the complexity of A, the only case of interest being that where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that $\mathcal{N} \models_x \Box B$. Then we know that $\|B\| \in N(x)$. Thus, by the inductive hypothesis we know that $\mathcal{M} \models_y \tau_{\Diamond\Box}(B)$ for all points $y \in \|B\|$. By the definition of R_N we also know that there is a point $\langle x, X\rangle$ (where $X = \|B\|^{\mathcal{N}}$) such that $R\langle x, X\rangle, y$ for all such points y. Thus we can see that $\mathcal{M}_N \models_{\langle x,X\rangle} \Box\tau_{\Diamond\Box}(B)$. Again by the definition of R_N we know that $Rx\langle x, X\rangle$ and thus that $\mathcal{M}_N \models_x \Diamond\Box\tau_{\Diamond\Box}(B)$.

For the 'if' direction suppose that $\mathcal{M}_N \models_x \Diamond\Box\tau_{\Diamond\Box}(B)$. Then there is a point $y \in R_N(x)$ such that $\mathcal{M}_N \models_y \Box\tau_{\Diamond\Box}(B)$. By the definition of R_N we know that all such points y are of the form $\langle x, X\rangle$ for some neighbourhood $X \in N(x)$. Letting $y = \langle x, X\rangle$ we know by the definition of R_N that for all points $z \in X$ that $R\langle x, X\rangle z$ and that $\mathcal{M}_N \models_z \tau_{\Diamond\Box}(B)$. By the inductive hypothesis this means that $\mathcal{N} \models_z B$ for all points $z \in X$ and that $X \in N(x)$. Thus we can see that $X \subseteq \|B\|$ and consequently by supplementation we know that $\|B\| \in N(x)$, from which it follows that $\mathcal{N} \models_x \Box B$. □

We are now in a position to show that we can faithfully embed **EM** into **K** using the translation $\tau_{\Diamond\Box}$.

Theorem 6.1.11. **EM** *is faithfully embedded into* **K** *by the translation* $\tau_{\Diamond\Box}$.

Proof. What we need to show is that the following holds for all formulas A.

$$A \in \mathbf{EM} \text{ if and only if } \tau_{\Diamond\Box}(A) \in \mathbf{K}. \tag{6.6}$$

For the 'only if' direction suppose that $\tau_{\Diamond\Box}(A) \notin \mathbf{K}$. Then there is a Kripke model $\mathcal{M} = \langle W, R, V\rangle$ and a point $x \in W$ such that $\mathcal{M} \not\models_x \tau_{\Diamond\Box}(A)$. By Theorem 6.1.8 it follows that $\mathcal{N}_R^+ \not\models_x A$. As $\mathcal{N}_R^+$ is supplemented and thus a model on a neighbourhood frame for **EM**, we can conclude that $A \notin \mathbf{EM}$.

For the 'if' direction suppose that $A \notin \mathbf{EM}$. Then there is a supplemented neighbourhood model $\mathcal{N} = \langle W, N, V\rangle$ and a point $x \in W$ such that $\mathcal{N} \not\models_x A$. By Theorem 6.1.10 we know that $\mathcal{M}_N \not\models_x \tau_{\Diamond\Box}(A)$ and, as this is a Kripke model, that $\tau_{\Diamond\Box}(A) \notin \mathbf{K}$. □

Theorem 6.1.12. *If, for some modality* ∇ *we have that* $\mathsf{S}(\nabla) = \mathbf{EM}$ *for some modal logic* S *then* $\mathsf{S}(\overline{\nabla}) = \mathbf{EM}$, *where* $\overline{\nabla}$ *is the dual of* ∇

Proof. Suppose that $\mathsf{S}(\nabla) = \mathbf{EM}$. Then as $\mathbf{EM}(\Diamond) = \mathbf{EM}$ we know that $\mathsf{S}(\nabla) = \mathbf{EM}(\Diamond)$. It is easy to verify that $\mathsf{S}'(\Diamond) = \mathsf{S}'' \iff \mathsf{S}''(\Diamond) = \mathsf{S}'$ and thus that $(\mathsf{S}(\nabla))(\Diamond) = \mathsf{S}(\overline{\nabla}) = \mathbf{EM}$ as desired. □

Let us say that a sequence $O_1 \dots O_n$ where each $O_i \in \{\Box, \Diamond\}$ is a *mixed linear modality* iff there are O_i, O_j such that $1 \leq i, j \leq n$ and $O_i = \Box$ and $O_j = \Diamond$. In view of the above theorem we know that, when we are concerned with the analogousness of mixed linear modalities in **K** we can assume that they are all of the form $O_1 \dots O_n \Diamond\Box O_{n+1} \dots O_{n+m}$. What we will now proceed to do is to show that all mixed linear modalities in **K** are analogous to $\Diamond\Box$, for which we need to the following model construction. For the following construction we will need to add, for each point $x \in W$ (where W is the set of worlds in the neighbourhood model $\mathcal{N}$) a set of new points $\{x_1^-, \dots, x_n^-\}$ for some (fixed) $n \in Nat$, as well as also adding the set $\{x_1^+, \dots, x_m^+\}$ again for some (fixed) $m \in Nat$. Let us call the set of all such new points W_n^- and W_m^+ respectively. Additionally let us define the following relations on the sets W_n^- and W_m^+:

$$R_n^- = \{\langle x_i^-, x_{i+1}^- \rangle | 1 \leq i < n\}. \tag{6.7}$$

$$R_m^+ = \{\langle y_j^+, y_{j+1}^+ \rangle | 1 \leq j < m\}. \tag{6.8}$$

We are now in a position to define the new model we will need.

Definition 6.1.13. Let $\mathcal{M}_N = \langle W_N, R_N, V \rangle$ be as in Definition 6.1.9. Let us define the new model $\mathcal{M}_n^m = \langle W_n^m, R_n^m, V \rangle$ for some fixed $m, n \in Nat$ as follows.

- $W_n^m := W_N \cup W_n^- \cup W_m^+$.
- $R_n^m := R_n^- \cup R_m^+ \cup \{\langle x, x_1^- \rangle | x \in W\} \cup \{\langle x_n^-, y \rangle | R_N xy \text{ and } x \in W\} \cup$
  $\{\langle x, y_1^+ \rangle | R_N xy \text{ and } y \in W\} \ \cup \{\langle y_m^+, y \rangle | y \in W\}$.

Theorem 6.1.14. *Let $\mathcal{M}_N$ be as in Definition 6.1.9, and $\mathcal{M}_n^m$ be constructed from it as per Definition 6.1.13. Then for all points $x \in W$, and all formulas A we have the following.*

$$\mathcal{M}_N \models_x \tau_{\Diamond\Box}(A) \textit{ if and only if } \mathcal{M}_n^m \models_x \tau(A).$$

Where $\tau(\Box A) = O_1 \dots O_n \Diamond\Box O_{n+1} \dots O_{n+m} \tau(A)$, for any mixed linear modality $O_1 \dots O_n \Diamond\Box O_{n+1} \dots O_{n+m}$.

Proof. By induction upon the complexity of A, the only case of interest being that where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that $\mathcal{M}_N \models_x \Diamond\Box\tau_{\Diamond\Box}(B)$. Then there is a points $\langle x, X\rangle$ such that $R_N x\langle x, X\rangle$ and $\mathcal{M}_N \models_{\langle x,X\rangle} \Box\tau_{\Diamond\Box}(B)$. This means that for all points z such that $R\langle x, X\rangle z$ we have that $\mathcal{M}_N \models_z \tau_{\Diamond\Box}(B)$. By the inductive hypothesis it follows then that $\mathcal{M}_n^m \models_z \tau(B)$. As $R_N\langle x, X\rangle z$ and $z \in W$ we know that for all such z, $(R_n^m)^{m+1}\langle x, X\rangle z$ and thus that $\mathcal{M}_n^m \models_{\langle x,X\rangle} \Box O_1 \dots O_m \tau(B)$. As $R_N x\langle x, X\rangle$ and $x \in W$ it follows that $(R_n^m)^{n+1}x, \langle x, X\rangle$ and thus that $\mathcal{M}_n^m \models_x O_1 \dots O_n \Diamond\Box O_{n+1} \dots O_{n+m}\tau(B)$.

For the 'if' direction suppose that $\mathcal{M}_n^m \models_x O_1 \dots O_n \Diamond\Box O_{n+1} \dots O_{n+m}\tau(A)$. Then as $x \in W$ we know that $(R_n^m)^{n+1}x\langle x, X\rangle$ for some point $\langle x, X\rangle \in W_N$ such that $\mathcal{M}_n^m \models_{\langle x,X\rangle} \Box O_{n+1} \dots O_{n+m}\tau(A)$. Thus we know that for all points z such that $(R_n^m)^{m+1}\langle x, X\rangle z$ that $\mathcal{M}_n^m \models_z \tau(B)$. By the inductive hypothesis we know that $\mathcal{M}_N \models_z \tau_{\Diamond\Box}(B)$. As $(R_n^m)^{m+1}\langle x, X\rangle z$ we know that $R_N\langle x, X\rangle z$ for all such points z, and thus that $\mathcal{M}_N \models_{\langle x,X\rangle} \Box\tau_{\Diamond\Box}(B)$. By the definition of R_N we also know that $R_N x\langle x, X\rangle$ and thus that $\mathcal{M}_N \models_x \Diamond\Box\tau_{\Diamond\Box}(B)$. □

Theorem 6.1.15. *For all mixed linear modalities* ∇ *we have that* $\mathbf{K}(\nabla) = \mathbf{EM}$.

Proof. As every modality is monotone in $\mathbf{K}$ it is clear that $\mathbf{EM} \subseteq \mathbf{K}(\nabla)$. To show the reverse inclusion suppose that $A \notin \mathbf{EM}$. By the completeness of $\mathbf{EM}$ w.r.t. supplemented neighbourhood models it follows that there is a supplemented neighbourhood model $\mathcal{N} = \langle W, N, V\rangle$ and a point $x \in W$ such that $\mathcal{N} \not\models_x A$. By Theorem 6.1.10 we know that $\mathcal{M}_N \not\models_x \tau_{\Diamond\Box}(A)$. Using Theorem 6.1.12 can assume without loss of generality that $\nabla = O_1 \dots O_n \Diamond\Box O_{n+1} \dots O_{n+m}$, and thus by Theorem 6.1.14 that $\mathcal{M}_n^m \not\models_x \tau(A)$. Thus as $\mathcal{M}_n^m$ is a model on a frame for $\mathbf{K}$ it follows that $\tau(A) \notin \mathbf{K}$ and thus that $A \notin \mathbf{K}(\nabla)$. □

Corollary 6.1.16. *For all mixed linear modalities* ∇ *and* ∇' *we have that* $\mathbf{K}(\nabla) = \mathbf{K}(\nabla')$.

6.1.2 Some Extensions of the Result

As we mentioned above, similar results to ours can be found in Kracht & Wolter [1999]. There it is shown that we can faithfully embed **EM** into bimodal **K** – the logic sometimes called $\mathbf{K} \times \mathbf{K}$ or $\mathbf{K}^2$ – using the translation $(\cdot)^D$ for which $(\Box A)^D = \Diamond_1 \Box_2 (A)^D$. The intuitive idea behind this translation is that $\Box_1$ is being used to quantify over neighbourhoods, and $\Box_2$ to quantify within the neighbourhoods. One of the more interesting results obtained there is the following. For any set of formulas Γ , we define $\mathbf{EM}{+_m}\Gamma$ to be the smallest monotonic modal logic containing Γ, and similarly $\mathbf{E}{+_e}\Gamma$ to be the smallest congruential modal logic containing Γ.

Theorem 6.1.17 (Kracht & Wolter [1999, p.109])**.** *For all formulas A we have the following:*

$$A \in \mathbf{EM} +_m \Gamma \textit{ if and only if } (A)^D \in \mathbf{K}^2 \oplus \Gamma^D.$$

Thus to every monotonic modal logic there corresponds at least one normal bimodal logic – namely the logic which the above theorem tells us it can be faithfully embedded into by $(\cdot)^D$. Providing a general result like this is not possible without resorting to more elaborate semantic structures (in this case general frames). In particular, without resorting to structures such that every monotonic modal logic is determined by some class of them. Perhaps more interesting, though, is the fact that the translation $\tau_{\Diamond\Box}$ only establishes a correspondence like that above between monotonic modal logics and quasi-normal modal logics (as opposed to normal modal logics, as above).

Theorem 6.1.18. *For all formulas A we have the following:*

$$A \in \mathbf{EM} +_m \Gamma \textit{ if and only if } \tau_{\Diamond\Box}(A) \in \mathbf{K} + \tau_{\Diamond\Box}(\Gamma).$$

Proof. For the 'if' direction suppose that $A \notin \mathbf{EM} +_m \Gamma$. Then there is a model $\mathcal{N} = \langle W, N, V \rangle$ for $\mathbf{EM} +_m \Gamma$, and a point $x \in W$ such that $\mathcal{N} \not\models_x A$. By Theorem 6.1.10 we know that the model $\mathcal{M}_N$ such that $\mathcal{M}_N \not\models_x \tau_{\Diamond\Box}(A)$.

Furthermore, by Theorem 6.1.10 it also follows that for all points $y \in W$ it follows that $\mathcal{M}_N \models_y \tau_{\Diamond\Box}(A)$. Letting $D = W$ it follows that $\langle W_N, R_N, D, V\rangle \not\models_x \tau_{\Diamond\Box}(A)$. It is easy to see that this is a model on a frame w.d.e. for $\mathbf{K} + \tau_{\Diamond\Box}(\Gamma)$ it follows that $\tau_{\Diamond\Box}(A) \notin \mathbf{K} + \tau_{\Diamond\Box}(\Gamma)$.

The 'only if' direction follows trivially. □

Thus, for example we are able to see that $\tau_{\Diamond\Box}$ faithfully embeds the monotonic modal logic **EMT** (i.e. $\mathbf{EM} +_m \mathbf{T}$) into $\mathbf{K} + \mathbf{B}$.

Thus is is quite easy to show that to every monotonic modal logic there corresponds a quasi-normal modal logic – the quasi normal modal logic into which it can be faithfully embedded by $\tau_{\Diamond\Box}$. One obvious question to ask is whether we can strengthen the + in the above theorem to $\oplus$ – that is to say, can every monotonic modal logic $\mathbf{EM} +_m \Gamma$ be faithfully embedded into the normal modal logic $\mathbf{K} \oplus \tau_{\Diamond\Box}(\Gamma)$.

To see that this is not the case consider what happens when we let $\Gamma = \{(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q), \Box\top\}$ – that is, what happens when we let the source logic of our translation be **K** itself. In this case the set $\tau_{\Diamond\Box}(\Gamma)$ is **K**-equivalent to the set $\{\mathbf{H}, \Diamond\top\}$.[7]

$$\mathbf{H}: \quad (\Diamond\Box p \wedge \Diamond\Box q) \rightarrow \Diamond\Box(p \wedge q).$$

Thus in order for this to hold we would have to be able to show that $\tau_{\Diamond\Box}$ faithfully embeds **K** into **KDH**. But it is easy to see that every extension of **KD** proves $\Box\Diamond\top$ (because **KD** does) which is just $\tau_{\Diamond\Box}(\Diamond\top)$ – a **K**-unprovable formula. As it happens even if we let $\Gamma = \{(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q), \Box\top, \Diamond\top\}$ then $\tau_{\Diamond\Box}(\Gamma)$ again is **K**-equivalent to $\{\mathbf{H}, \Diamond\top\}$. In this case we fare no better, this time the problem formula being recorded in Proposition 3.2 of Humberstone [2006]. As a consequence, we will have to content ourselves with Theorem 6.1.18 as it stands, and leave the project of seeing whether it can be generalized further to one side.

[7] We have already encountered the logic **KDH** in connection with the $\tau_{\Diamond\Box}$ translation on p.73. For further information the reader should consult Humberstone [2006].

Theorem 6.1.18 can be interpreted as telling us that the translation $\tau_{\diamond\Box}$ induces a map f whose domain is the set of all monotonic extensions of **EM**, and whose co-domain is some subset of the extensions of **K**. One obvious line of investigation then is to determine some of the properties of this mapping f – whose intended definition is as follows.

$$f(\mathbf{EM}+\Gamma) = \mathbf{K} + \tau_{\diamond\Box}(\Gamma). \tag{6.9}$$

It is easy to show that f is $\cap$-semilattice morphism. Firstly we might want to know what the maximal "quasi-normal companions" are. That is, what are the greatest elements (under $\subseteq$) of the co-domain of f. Henceforth, in the interests of brevity, we will denote the co-domain of f as Δ. As a first step towards characterizing the maximal elements of Δ we will recall the following result from Kracht & Wolter [1999].

Proposition 6.1.19. *Every consistent monotonic modal logic is a sub-logic of the logic determined by one of the following neighbourhood frames.*

$$\mathfrak{F}_1 = \langle\{0\},\emptyset\rangle \quad \mathfrak{F}_2 = \langle\{0\},\langle 0,\{\emptyset\}\rangle\rangle \quad \mathfrak{F}_3 = \langle\{0\},\langle 0,\{\{0\}\}\rangle\rangle$$

The neighbourhood frames $\mathfrak{F}_2$ and $\mathfrak{F}_3$ can be quite easily shown to determine the logics **KVer** and **KT!**. The remaining frame, $\mathfrak{F}_1$ determines the logic $\mathbf{EM}+\Diamond p$. Consider now the following three relational frames.

$$\begin{aligned}
\mathfrak{F}_{1N} &= \langle\{0\},\emptyset\rangle \\
\mathfrak{F}_{2N} &= \langle\{0,\langle 0,\emptyset\rangle\},\{\langle 0,\langle 0,\emptyset\rangle\rangle\}\rangle \\
\mathfrak{F}_{3N} &= \langle\{0,\langle 0,\{0\}\rangle\},\{\langle 0,\langle 0,\{0\}\rangle\rangle,\langle\langle 0,\{0\}\rangle,0\rangle\}\rangle.
\end{aligned}$$

It is quite easy to see that these frames result from the neighbourhood frames $\mathfrak{F}_i$ above by applying the construction in Definition 6.1.9. Moreover, it is easy to see that the logics determined by the above frames when the distinguished element is 0 are the logics into which those determined by the neighbourhood frames $\mathfrak{F}_i$ can be faithfully embedded by $\tau_{\diamond\Box}$.

Proposition 6.1.20. *For all formulas A we have the following.*

(i)	$A \in \mathbf{EM} + \Diamond p$	*if and only if* $\tau_{\Diamond\Box}(A) \in \mathbf{K} + \Box\Diamond p$
(ii)	$A \in \mathbf{KVer}$	*if and only if* $\tau_{\Diamond\Box}(A) \in \mathbf{K} + \Diamond\Box p$
(iii)	$A \in \mathbf{KT!}$	*if and only if* $\tau_{\Diamond\Box}(A) \in \mathbf{K} + \{\mathbf{H}, \Diamond\Box\top, \Diamond\Box p \leftrightarrow p\}$

It bears noting that, even if the logic $\mathbf{K} + \Diamond\Box p$ is the same as the logic $\mathbf{K} + \mathbf{H} + \Diamond\Box p$ – all being instances of the tautology $A \to \top$. As all of the source logics listed in the above Proposition are all and only the Post-complete monotonic modal logics, it follows that the logics above exhaust the elements of Δ.

6.2 Translations in Congruential Modal Logics

In Kracht & Wolter [1999] it is shown that we can faithfully embed **E** into trimodal **K** using the modal-to-modal translation $(\cdot)^F$ for which $(\Box A)^F = \Diamond_1(\Box_2(A)^F \wedge \Box_3\neg(A)^F)$. From a semantic point of view the idea is to think of $\Diamond_1$ as quantifying over neighbourhoods, $\Box_2$ as quantifying within neighbourhoods, and $\Box_3$ as quantifying over their complements – thus allowing us to mimic the truth conditions for $\Box$-formulas within a neighbourhood model. In Gasquet & Herzig [1996] it is noted that we can simplify this translation to one which faithfully embeds **E** in bimodal **K** by making $\Box_1$ and $\Box_2$ the same operator – thus making the new translation be such that $(\Box A)^{F'} = \Diamond_1(\Box_1(A)^{F'} \wedge \Box_2\neg(A)^{F'})$. One might then wonder whether a further simplification is possible, allowing us to embed **E** faithfully in monomodal **K**. As it happens the simplification cannot be of the same nature as that given in Gasquet & Herzig [1996] – as the translation $(\cdot)^{F''}$ for which $(\Box A)^{F''} = \Diamond(\Box(A)^{F''} \wedge \Box\neg(A)^{F''})$ fails to faithfully embed **E** into **K**, there being **K**-provable formulas of the form $(A)^{F''}$ for which A is not **E**-provable.[8] Consequently we will have to look elsewhere for a translation

[8]For example, let A be the formula "$\Box p \leftrightarrow \Box\neg p$", whose $(\cdot)^{F''}$-translation is provable in every congruential modal logic.

which faithfully embeds **E** into monomodal **K**.

Translations of this sort are notoriously hard to find, the ones which are extractable from the literature being extremely complex. Before going on to give our simple translation we will first in this section survey some of the potential translations which might spring to mind in order to give the reader a feel for the difficulty of finding such translations.

6.2.1 Thomason's Translation

In Kracht & Wolter [1999], building on some work done in the 1970s by S.K. Thomason (Thomason [1974; 1976]), we are presented with a general account of how to faithfully embed all bimodal normal modal logics into normal monomodal modal logics. The account given there, while being quite novel in many ways, is somewhat lacking in details in some areas – which we will endeavour to fill in here. The strategy implied by the title of Kracht & Wolter [1999] is a two step process for faithfully embedding arbitrary modal logics into normal monomodal logics. The first step is to show that we can faithfully embed every modal logic from a given class of modal logics (i.e. the congruential modal logics) into the extensions of a particular normal multi-modal logic. In the second step we then apply Thomason's translation to faithfully embed the extensions of this normal multi-modal logic into the extensions of a particular normal monomodal logic.

Kracht and Wolter's treatment of the second step of this process is somewhat unfulfilling, as a great number of the results of the first type they address require us to embed the relevant class of modal logics into normal extensions of *tri*-modal **K**, while their discussion of the Thomason translation only deals with the bimodal case! What we will do in this section is look at this trimodal version of the Thomason result, focusing in particular on how it applies to the translation $(\cdot)^F$ which embeds **E** into $\mathbf{K}^3$. In particular we will compare this translation to the translation of **E** into monomodal **K** we get via the embedding $(\cdot)^{F'}$ of **E** into $\mathbf{K}^2$, and it is this translation we will discuss first. Before doing this though we will first need to introduce the bimodal version of the Thomason translation T_2.[9]

In discussing the bimodal version of the Thomason translation, T_2, we will (in the interests of readability) make use of the following abbreviations: $\mathsf{t} = \Box\bot$, $\mathsf{w} = \Diamond\Box\bot$ and $\mathsf{b} = \neg\mathsf{t} \wedge \neg\mathsf{w}$.[10] The *bimodal Thomason translation* – T_2 – is the following function which maps formulas of the language of bimodal logic to formulas of the language of monomodal logic.

$$\begin{aligned} \mathsf{T}_2(p_i) &= p_i \\ \mathsf{T}_2(A \wedge B) &= \mathsf{T}_2(A) \wedge \mathsf{T}_2(B) \\ \mathsf{T}_2(\neg A) &= \neg \mathsf{T}_2(A) \\ \mathsf{T}_2(\Box_1 A) &= \Box(\mathsf{w} \to \mathsf{T}_2(A)) \\ \mathsf{T}_2(\Box_2 A) &= \Box(\mathsf{b} \to \Box(\mathsf{b} \to \Box(\mathsf{w} \to \mathsf{T}_2(A)))). \end{aligned}$$

We will occasionally use $\Box_w A$ as an abbreviation for $\Box(\mathsf{w} \to A)$ and $\Box_b A$ as an abbreviation for $\Box(\mathsf{b} \to A)$ and $\Box_t A$ as an abbreviation for $\Box(\mathsf{t} \to A)$. It bears noting that this translation is very different from the translation – which we will dub T_2' given in Thomason [1974, p.550] – which is exactly like T_2 except that $\mathsf{T}_2'(p_i) = \mathsf{w} \wedge p_i$ and $\mathsf{T}_2'(\neg A) = \mathsf{w} \wedge \neg \mathsf{T}_2'(A)$. This translation is also different from the one given in Kracht [1999, p.398] – T_2'' – which is identical to T_2 except that $\mathsf{T}_2''(\neg A) = \mathsf{w} \wedge \neg \mathsf{T}_2''(A)$. [11]

[9] We will adopt the convention here of using T_i to designate the Thomason-style translation which faithfully embeds $\mathbf{K}^i$ into $\mathbf{K}$.

[10] These labels are taken from Kracht & Wolter [1999], where they are mnemonic for 'terminal', 'white' and 'black' respectively – Kracht and Wolter using $\Box$ and $\blacksquare$ for what we're calling $\Box_1$ and $\Box_2$.

[11] This translation does not do the work which Kracht sets for it – the crucial result (Proposition 6.6.14 of Kracht [1999]) being incorrect. Consider the model $\mathcal{M} = \langle \{x, y\}, \{\langle x, y\rangle\}, \emptyset, V\rangle$ where $V(p) = \{x\}$. Let $V'(p) = \{x^\circ, x^\bullet\}$ – then V' is a valuation such that $V'(p) \cap W^\circ = (V(p))^\circ$. It is easy to see that $\mathcal{M}^s \models_{x^\circ} \mathsf{w} \wedge \Diamond p$ (alias. $\mathsf{T}_2''(\Diamond_1 p)$) while $\mathcal{M} \not\models_x \Diamond_1 p$ – this being a false instance of Proposition 6.6.14 in Kracht [1999]. The incorrectness of this crucial result makes all the results there concerning the Thomason-style translation T_2'' incorrect. This problem can be avoided if we alter the translation so that

Definition 6.2.1. Let $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ be a bimodal model, and $\{w, b, t\}$ be new points not in W, with x^w and x^b being abbreviations for $\langle x, w\rangle$ and $\langle x, b\rangle$. Then let us construct a new model $\mathcal{M}^{sim} = \langle W^{sim}, R^{sim}, V^{sim}\rangle$ as follows.[12]

- $W^{sim} := W \times \{w, b\} \cup \{t\}$
- $R_w := \{\langle x^w, y^w\rangle | R_1 xy\}$
- $R_b := \{\langle x^b, y^b\rangle | R_2 xy\}$
- $R_* := \{\langle x^w, t\rangle, \langle x^w, x^b\rangle, \langle x^b, x^w\rangle | x \in W\}$
- $R^{sim} := R_w \cup R_b \cup R_*$
- $V^{sim}(p_i) := V(p_i) \times \{w\}$.

Theorem 6.2.2 (Kracht & Wolter [1999, p.122]). *Let $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ be a bimodal model, and $\mathcal{M}^{sim}$ be as above. Then for all bimodal formulas A and all points $x \in W$ we have the following.*

$$\mathcal{M} \models_x A \textit{ if and only if } \mathcal{M}^{sim} \models_{x^w} \mathsf{T}_2(A).$$

The second step of the Kracht & Wolter Reduction (of normal bimodal logics to normal monomodal logics) is to then, using the above result, show that we can faithfully embed any given normal bimodal logic S into the logic determined by a particular class of frames (the Sim-Frames for S). Some syntactic results concerning the bimodal Thomason translation can be found in Kracht & Wolter [1999, p.122].

$\mathsf{T}_2''(p_i) = \mathsf{w} \wedge p_i$ – making it more closely resemble the translation T_2' from Thomason [1974]. This (corrected) translation appears in Kracht & Wolter [1997]. This example makes it clear that the problem is with the combination variable-fixedness and that particular way of translating the modal operators.

[12] We call this model $\mathcal{M}^{sim}$ in reference to the fact that what we are calling the Thomason translation is called by others, such as Kracht & Wolter [1999] the Thomason *simulation*.

This result is used in conjunction with the following result in order to give our (first) faithful embedding of **E** into **K**.

Theorem 6.2.3 (Gasquet & Herzig [1996, p.307]).

$$A \in \mathbf{E} \textit{ if and only if } (A)^{F'} \in \mathbf{K}_2.$$

Theorem 6.2.4.

$$A \in \mathbf{E} \textit{ if and only if } \mathsf{T}_2((A)^{F'}) \in \mathbf{K}.$$

Proof. For the 'if' direction suppose that $A \notin \mathbf{E}$. Then by Theorem 6.2.3 it follows that $(A)^{F'} \notin \mathbf{K}_2$. So there is a model $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ and a point $x \in W$ such that $\mathcal{M} \not\models_x A^{F'}$. By Theorem 6.2.2 it then follows that $\mathcal{M}^{sim} \not\models_{x^w} \mathsf{T}_2((A)^{F'})$. Thus, as **K** is complete w.r.t. the class of all Kripke models it follows then that $\mathsf{T}_2((A)^{F'}) \notin \mathbf{K}$ as desired.

The 'only if' direction follows by induction upon the length of derivations of A, the only case of interest coming in the inductive step where A follows from the congruentiality of $\Box$. But as all contexts are congruential in **K** it follows that if $A \leftrightarrow B \in \mathbf{K}$ then $C(A) \leftrightarrow C(B) \in \mathbf{K}$ for the case where $C(p) = \mathsf{T}_2(\Box p)^{F'}$). $\Box$

This gives us a translation τ where $\tau(\Box A) =$

$$\Diamond(\Diamond\Box\bot \wedge [\Box(\Diamond\Box\bot \rightarrow \tau(A)) \wedge \Box((\neg\Diamond\Box\bot \wedge \neg\Box\bot) \rightarrow \Box((\neg\Diamond\Box\bot \wedge \neg\Box\bot) \rightarrow \Box(\Diamond\Box\bot \rightarrow \neg\tau(A))))]).$$

As we can see above, the context which we are using here to translate $\Box$ is quite unwieldy, transforming formulas of modal degree n to formulas of modal degree $6n$. It would be nice if we could find a translation of a lower degree of complexity in this sense, the search for which we will delay for a moment to look first at what the trimodal Thomason simulation looks like, and how it bears on the particular problem of embedding **E** into **K**.

We will consider our trimodal language as having three modal operators $\Box_0$, $\Box_1$ and $\Box_2$. As with the bimodal Thomason translation we will also need a number of constants and derived modal operators. Let $\mathsf{a_3} = \Diamond(\Diamond\Box\bot \wedge \Diamond\Diamond\Box\bot)$, $\mathsf{b_3} = \Diamond(\Diamond\Box\bot \wedge \neg\Diamond\Diamond\Box\bot)$ and $\mathsf{c_3} = \Diamond\Box\bot \wedge \Diamond\Diamond\Diamond\Box\bot$. Furthermore let $\Box_a A = \Box(\mathsf{a_3} \to A)$, $\Box_b A = \Box(\mathsf{b_3} \to A)$ and $\Box_c A = \Box(\mathsf{c_3} \to A)$. Then consider the following translation from the language of tri-modal logic, to that of monomodal logic.

$$
\begin{array}{rcl}
\mathsf{T}_3(p_i) & = & p_i \\
\mathsf{T}_3(A \wedge B) & = & \mathsf{T}_3(A) \wedge \mathsf{T}_3(B) \\
\mathsf{T}_3(\neg A) & = & \neg \mathsf{T}_3(A) \\
\mathsf{T}_3(\Box_0 A) & = & \Box_a \Box_a \Box_c \mathsf{T}_3(A) \\
\mathsf{T}_3(\Box_1 A) & = & \Box_b \Box_b \Box_c \mathsf{T}_3(A) \\
\mathsf{T}_3(\Box_2 A) & = & \Box_c \mathsf{T}_3(A)
\end{array}
$$

We will now discuss the trimodal Thomason translation from a model theoretic perspective. Given a model $\mathcal{M} = \langle W, R_0, R_1, R_2, V\rangle$ let us construct a new model $\mathcal{M}^{sim3} = \langle W^{sim3}, R, V^{sim3}\rangle$ where:

$$
\begin{array}{rcl}
W^{sim3} & = & W \times \{0,1,2\} \cup \{0,1,2\} \\
R & = & \{\langle (x,i),(y,i)\rangle | R_i xy\} \cup \{\langle (x,i),(x,j)\rangle | i \neq j\}\} \\
 & & \cup \{\langle (x,i), i\rangle | x \in W\} \cup \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle\}. \\
V^{sim3}(p_i) & = & V(p_i) \times \{2\}.
\end{array}
$$

Lemma 6.2.5. *Let $\mathcal{M} = \langle W, R_1, R_2, R_3, V\rangle$, and defined $\mathcal{M}^{sim}$ as above. Then for all points $x \in W$ we have the following.*

$$
\begin{array}{lll}
(i)\ \mathcal{M}^{sim} \models_{(x,i)} \mathsf{a_3} & \textit{if and only if} & i = 0 \\
(ii)\ \mathcal{M}^{sim} \models_{(x,i)} \mathsf{b_3} & \textit{if and only if} & i = 1 \\
(iii)\ \mathcal{M}^{sim} \models_{(x,i)} \mathsf{c_3} & \textit{if and only if} & i = 2
\end{array}
$$

Proposition 6.2.6. *For all formulas A and all points $x \in W$ we have the following:*

$$\mathcal{M} \models_x A \textit{ if and only if } \mathcal{M}^{sim3} \models_{(x,2)} \mathsf{T}_3(A).$$

Proof. By induction upon the complexity of A, the case of interest being that where (i) $A = \Box_2 B$ and (ii) where $A = \Box_0 B$ (the case of $A = \Box_1 B$ being identical in detail).

For (i) suppose that $\mathcal{M} \models_x \Box_2 B$. Then we know that for all points $y \in R_2(x)$ that $\mathcal{M} \models_y B$. By the inductive hypothesis it follows then that $\mathcal{M}^{sim3} \models_{(y,2)} \mathsf{T}_3(B)$. By Lemma 6.2.5 we know that all these points also verify c_3, and additionally we can see that these are all the points in $R((x,2))$ which verify c_3 – allowing us to conclude that $\mathcal{M}^{sim3} \models_{(x,2)} \Box_c \mathsf{T}_3(B)$. Suppose now that $\mathcal{M}^{sim3} \models_{(x,2)} \Box_c \mathsf{T}_3(B)$. Then we know that for all points $(y,i) \in R((x,2))$ which verify c_3 also verify $\mathsf{T}_3(B)$. By the induction hypothesis then we know that $\mathcal{M} \models_y B$ for all such points y. By Lemma 6.2.5 it follows that $i = 2$ for all such points (y,i), and thus that $R_2 xy$. It follows by the construction of $\mathcal{M}^{sim3}$ that these are precisely the points $y \in R_2(x)$, and thus that $\mathcal{M} \models_x \Box_2 B$.

For (ii) suppose that $\mathcal{M} \models_x \Box_0 B$. Then we know that for all points $y \in R_0(x)$ that $\mathcal{M} \models_y B$. By the inductive hypothesis it follows then that $\mathcal{M}^{sim3} \models_{(y,2)} \mathsf{T}_3(B)$. As the only point verifying c_3 which is R-accessible from each of the points $(y,0)$ is the point $(y,2)$ it follows that $\mathcal{M}^{sim3} \models_{(y,0)} \Box_c \mathsf{T}_3(B)$. As $R_0 xy \iff R(x,0)(y,0)$ it then follows that $\mathcal{M}^{sim3} \models_{(x,0)} \Box_a \Box_c \mathsf{T}_3(B)$, and as this is the only a_3-verifying point R-accessible to $(x,2)$, that $\mathcal{M}^{sim3} \models_{(x,2)} \Box_a \Box_a \Box_c \mathsf{T}_3(B)$. Suppose now that $\mathcal{M}^{sim3} \models_{(x,2)} \mathsf{T}_3(\Box_0 B)$, that is $\mathcal{M}^{sim3} \models_{(x,2)} \Box_a \Box_a \Box_c \mathsf{T}_3(B)$. Then we know that for all a_3-verifying points y in $R((x,2))$ that $\mathcal{M}^{sim3} \models_y \Box_a \Box_c \mathsf{T}_3(B)$. As the only such point is $(x,0)$ it follows then that $\mathcal{M}^{sim3} \models_{(x,1)} \Box_a \Box_c \mathsf{T}_3(B)$. So for all a_3-verifying points z in $R((x,0))$ it follows that $\mathcal{M}^{sim3} \models_z \Box_c \mathsf{T}_3(B)$. By the construction of $\mathcal{M}^{sim3}$ we can see that these are the points $(z,0)$ for points $z \in R_0(x)$. So, by the fact that all of these points verify $\Box_c \mathsf{T}_3(B)$, it follows that for all such points $(z,0)$ that $\mathcal{M}^{sim3} \models_{(z,0)} \mathsf{T}_3(B)$. By the induction hypothesis it follows that $\mathcal{M} \models_z B$. As these points z are all the points R_0-accessible from x it follows then that $\mathcal{M} \models_x \Box_0 B$.

$\Box$

Applying this to the embedding of **E** into $\mathbf{K}^3$ ($(\cdot)^F$) we get a translation where $\tau(\Box A) =$

$$\Diamond(\Box[\Diamond(\Diamond\Box\bot \wedge \Diamond\Diamond\Box\bot) \to \Box(\Diamond(\Diamond\Box\bot \wedge \Diamond\Diamond\Box\bot) \to \Box((\Diamond\Box\bot) \to \tau(A)))]\wedge$$
$$\Box[\Diamond(\Diamond\Box\bot \wedge \neg\Diamond\Diamond\Box\bot) \to \Box(\Diamond(\Diamond\Box\bot \wedge \neg\Diamond\Diamond\Box\bot) \to$$
$$\Box((\Diamond\Box\bot) \to \neg\tau(A)))] \wedge \Diamond\Box\bot \wedge \Diamond\Diamond\Diamond\Box\bot).$$

Proposition 6.2.7. *For all formulas A we have the following:*

$$A \in \mathbf{E} \textit{ if and only if } \tau(A) \in \mathbf{K}.$$

Proof. For the 'only if' direction suppose $A \notin \mathbf{E}$. Then we know that $(A)^F \notin \mathbf{K}^3$. So there is a model $\mathcal{M} = \langle W, R_0, R_1, R_2, V\rangle$ and a point $x \in W$ such that $\mathcal{M} \not\models_x (A)^F$. So by the above proposition we know that $\mathcal{M}^{sim3} \not\models_{(x,0)} \mathsf{T}_3((A)^F)$ and thus, as **K** is complete w.r.t the class of all Kripke models, that $\mathsf{T}_3((A)^F) \notin \mathbf{K}$.

The 'if' direction follows from the fact that all contexts are congruential in **K**. $\Box$

This translation translates modal formulas of degree n into formulas of degree $7n$. So it would appear, as one would hope, that in simplifying the translation by identifying $\Box_1$ and $\Box_2$ – as we are allowed to do when our source logic is **E**, but not necessarily when we are considering one of its extensions, as noted in Gasquet & Herzig [1996] – we end up with a translation which produces formulas of a lower complexity. What we will now go on to look at are the prospects for further simplification in respect of modal degree, ending up with a simpler translation which does not require us to take detours through multi-modal logic.

6.2.2 'Detours along the Road'

Before giving our simplified translation of **E** into **K** we will first give some 'detours' which we took in getting to our simplified translation, which are both of independent interest, and also show some of the complexities involved with embedding **E** into **K**. Firstly, consider the translation τ_{∇_1} which replaces all occurrences of $\Box A$ with the following.

$$\nabla_1(A) = \Diamond(\Diamond\Box A \wedge \Diamond\Box\neg A).$$

What we will now show is that this translation faithfully embeds the modal logic $\mathbf{E}+_e\Box p \leftrightarrow \Box\neg p$ into **K**. It is worth noting that we could equally well have axiomatized this logic as $\mathbf{E}+_e\Box p \rightarrow \Box\neg p$ or $\mathbf{E}+_e\Box\neg p \rightarrow \Box p$ – as these three formulas are all inter-deducible in **E** by simple applications of the rule of replacement of equivalents. These second two formulas are commonly used in axiomatizations of modal logics of non-contingency – reflecting the fact that if it is contingent whether p is true, it is also contingent whether p is false.[13]

Lemma 6.2.8. *For all formulas A if $A \in \mathbf{E} +_e \Box p \leftrightarrow \Box\neg p$ then $\tau_{\nabla_1}(A) \in \mathbf{K}$.*

Proof. By induction upon the length of derivations of A.[14] For the basis case A is an axiom, the only case of interest being that where A is $\Box p \leftrightarrow \Box\neg p$. The τ_{∇_1}-translation of this can be shown to be **E**-provable (and hence **K**-provable) as follows.

[13]For more information on contingency and non-contingency logics see Humberstone [1995] and Kuhn [1995].

[14]Here we are thinking of the axiomatization of $\mathbf{E}+_e\Box p \leftrightarrow \Box\neg p$ as being the one we get by adding the rule of **RE** as a new rule, and $\Box A \leftrightarrow \Box\neg A$ as a new axiom schemata to a standard axiomatization of classical propositional logic with modus ponens as its only rule.

(1)	$\Diamond(\Diamond\Box p \wedge \Diamond\Box\neg p) \leftrightarrow \Diamond(\Diamond\Box p \wedge \Diamond\Box\neg p)$	TF.
(2)	$(\Diamond\Box p \wedge \Diamond\Box\neg p) \leftrightarrow (\Diamond\Box\neg p \wedge \Diamond\Box p)$	TF.
(3)	$\Diamond(\Diamond\Box p \wedge \Diamond\Box\neg p) \leftrightarrow \Diamond(\Diamond\Box\neg p \wedge \Diamond\Box p)$	$(2), \Diamond - \mathbf{RE}$
(4)	$\Diamond(\Diamond\Box p \wedge \Diamond\Box\neg p) \leftrightarrow \Diamond(\Diamond\Box\neg p \wedge \Diamond\Box p)$	$(1),(3) TF$.

For the inductive step suppose that we can derive A by applying some rule of inference to some formulas B_i $(i < n)$. The case of modus ponens is trivial – leaving us only with the case where the rule in question is **RE**. Suppose then that $B \leftrightarrow C \in \mathbf{E} + \Box p \leftrightarrow \Box\neg p$ and that $A = \Box B \leftrightarrow \Box C$. By the induction hypothesis we know that $\tau_{\nabla_1}(B \leftrightarrow C) \in \mathbf{K}$ – which as we have a modal translation means that $\tau_{\nabla_1}(B) \leftrightarrow \tau_{\nabla_1}(C) \in \mathbf{K}$, and thus by (RE) and the commutativity of $\wedge$ it follows that $\tau_{\nabla_1}(\Box B \leftrightarrow \Box C) \in \mathbf{K}$.

□

Definition 6.2.9. Given a neighbourhood model $\mathcal{N} = \langle W, N, V\rangle$ construct a Kripke model $\mathcal{M}_{EN} = \langle W_{EN}, R_{EN}, V\rangle$ as follows.

- $W_{EN} := W \cup \{a_{x,X}, \langle x, X\rangle, \langle x, -X\rangle | X \in N(x)\}$.
- $R_{EN} := \{\langle x, a_{x,X}\rangle | X \in N(x)\}$
  $\cup\{\langle a_{x,X}, \langle x, X\rangle\rangle, \langle a_{x,X}, \langle x, -X\rangle\rangle | X \in N(x)\}$
  $\cup\{\langle\langle x, X\rangle, y\rangle | y \in X\}$
  $\cup\{\langle\langle x, -X\rangle, y\rangle | y \in W \setminus X\}$.

The above model construction is quite similar to the one used in Definition 6.1.9 except that we have added the extra points $a_{x,X}$ (for each neighbourhood $X \in N(x)$) which are R_{EN}-related to one point which accesses all points in X and another point which accesses all points in the complement of X.

Proposition 6.2.10. $\mathbf{E}+\Box p \leftrightarrow \Box\neg p$ *is determined by the class of all neighbourhood frames whose neighbourhood sets are closed under complements. That is* *to say, the class of all neighbourhood frames $\langle W, N \rangle$ such that if $X \in N(x)$ then $W \setminus X \in N(x)$.*

Proposition 6.2.11. *Let $\mathcal{N} = \langle W, N, V \rangle$ be a neighbourhood model which is closed under complements. Then for all all formulas A and points $x \in W$ we have the following:*

$$\mathcal{N} \models_x A \textit{ if and only if } \mathcal{M}_{EN} \models_x \tau_{\nabla_1}(A).$$

Proof. By induction upon the complexity of A – the only case of interest being that in the inductive step where $A = \Box B$ for some formula B.

For the 'if' direction suppose that $\mathcal{N} \models_x \Box B$. Then we know that $X \in N(x)$ where $X = \|B\|$. By the inductive hypothesis we can thus reason that – for all points $y \in X$ – $\mathcal{M}_{EN} \models_y \tau_{\nabla_1}(B)$. As $X = R_{EN}(\langle x, X \rangle)$ we know then that $\mathcal{M}_{EN} \models_{\langle x,X \rangle} \Box\tau_{\nabla_1}(B)$. As these points are all of the points in $\mathcal{N}$ which verify B we thus know that for all points $z \in W \setminus \|B\|$ that $\mathcal{N} \models_x \neg B$. Thus by the inductive hypothesis we can reason that – for all such points $z \in W \setminus X$ – that $\mathcal{M}_{EN} \models_z \neg\tau_{\nabla_1}(B)$. As $R_{EN}(\langle x, -X \rangle) = W \setminus X$ we can conclude that $\mathcal{M}_{EN} \models_{\langle x,-X \rangle} \Box\neg\tau_{\nabla_1}(B)$. As $R_{EN}(a_{x,X}) = \{\langle x, X \rangle, \langle x, -X \rangle\}$ we can see that $\mathcal{M}_{EN} \models_{a_{x,X}} \Diamond\Box\tau_{\nabla_1}(B) \wedge \Diamond\Box\neg\tau_{\nabla_1}(B)$, and consequently that $\mathcal{M}_{EN} \models_x \Diamond(\Diamond\Box\tau_{\nabla_1}(B) \wedge \Diamond\Box\neg\tau_{\nabla_1}(B))$ as desired.

For the 'only if' direction suppose that $\mathcal{M} \models_x \Diamond(\Diamond\Box\tau_{\nabla_1}(B) \wedge \Diamond\Box\neg\tau_{\nabla_1}(B))$. Then we know that there is a point $a_{x,X} \in R_{EN}(x)$ such that $\mathcal{M}_{EN} \models_{a_{x,X}} \Diamond\Box\tau_{\nabla_1}(B) \wedge \Diamond\Box\neg\tau_{\nabla_1}(B)$. This means that either $\mathcal{M}_{EN} \models_{\langle x,X \rangle} \Box\tau_{\nabla_1}(B)$ or $\mathcal{M}_{EN} \models_{\langle x,X \rangle} \Box\neg\tau_{\nabla_1}(B)$. By the inductive Hypothesis this means that either $\|B\| \in N(x)$ or $\|\neg B\| \in N(x)$, from which it follows that $\mathcal{N} \models_x \Box B$ or $\mathcal{N} \models_x \Box\neg B$. In the second case we can simply appeal to the appropriate substitution instance of $\Box p \leftrightarrow \Box\neg p$ to conclude that $\mathcal{N} \models_x \Box B$ as desired. □

Theorem 6.2.12. *For all formulas A we have the following.*

$$A \in \mathbf{E} + \Box p \leftrightarrow \Box\neg p \textit{ if and only if } \tau_{\nabla_1}(A) \in \mathbf{K}.$$

Proof. The 'only if' direction follows via appeal to Lemma 6.2.8. For the 'if' direction suppose that $A \notin \mathbf{E} + \Box p \leftrightarrow \Box\neg p$. Then by Proposition 6.2.10 there is a neighbourhood model $\mathcal{N} = \langle W, N, V \rangle$ which is closed under complements, and a point $x \in W$ such that $\mathcal{N} \not\models_x A$. By Proposition 6.2.11 it follows that $\mathcal{M}_{EN} \not\models_x \tau_{\nabla_1}(A)$, and as this is a model on a frame for $\mathbf{K}$ that $\tau_{\nabla_1}(A) \notin \mathbf{K}$. □

The second translation we will consider is the one which replaces all occurrences of $\Box A$ with $\boxed{\top} A = \Box\top \to \Box\tau_{\boxed{\top}}(A)$. This translation can be quite easily shown to faithfully embed **EN** into **E**.

Lemma 6.2.13. *For all formulas A if $A \in \mathbf{EN}$ then $\tau_{\boxed{\top}}(A) \in \mathbf{E}$.*

Proof. By induction upon the length of derivations of A.[15] For the basis case A is an axiom, the only case of interest being that where A is $\Box\top$. In this case we have that $\tau_{\boxed{\top}}(\Box\top) = \Box\top \to \Box\top$ – which is provable in **E**.

For the inductive step suppose that we get A by applying some rule of inference to some formulas B_i. The case of modus ponens is trivial – leaving us only with the case where the rule in question is (RE). Suppose then that $B \leftrightarrow C \in \mathbf{EN}$ and that $A = \Box B \leftrightarrow \Box C$. By the induction hypothesis we know that $\tau_{\boxed{\top}}(B \leftrightarrow C) \in \mathbf{E}$ – which as we have a modal-to-modal translation means that $\tau_{\boxed{\top}}(B) \leftrightarrow \tau_{\boxed{\top}}(C) \in \mathbf{E}$.

(1)	$\tau_{\boxed{\top}}(B) \leftrightarrow \tau_{\boxed{\top}}(C)$	IH
(2)	$\Box\tau_{\boxed{\top}}(B) \leftrightarrow \Box\tau_{\boxed{\top}}(C)$	$(1), (RE)$
(3)	$(\Box\top \to \Box\tau_{\boxed{\top}}(C)) \leftrightarrow (\Box\top \to \Box\tau_{\boxed{\top}}(B))$	$(2), TF$

Thus we have $\tau_{\boxed{\top}}(\Box C \leftrightarrow \Box B) \in \mathbf{E}$ and the result follows. □

[15] Again we take as understood the axiomatization with **RE** as our only modal rule, and $\Box\top$ as our only modal axiom – the non-modal basis being one with modus ponens as the sole rule.

Theorem 6.2.14 (Chellas [1980]). **EN** *is determined by the class of all neighbourhood frames whose neighbourhood sets contains the unit. That is to say, the class of all neighbourhood frames* $\langle W, N\rangle$ *for which* $W \in N(x)$ *for all* $x \in W$.

Proposition 6.2.15. *Let* $\mathcal{N} = \langle W, N, V\rangle$ *be a neighbourhood frame whose neighbourhood function contains the unit. Then for all formulas* A *and all points* $x \in W$ *we have the following.*

$$\mathcal{N} \models_x A \textit{ if and only if } \mathcal{N} \models_x \tau_{\boxed{\top}}(A).$$

Proof. By induction upon the complexity of A – the only case of interest being that in the inductive step where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that $\mathcal{N} \models_x \Box B$. Then we know that $\|B\| \in N(x)$. By the inductive hypothesis this means that $\|\tau_{\boxed{\top}}(B)\| \in N(x)$, and thus that $\mathcal{N} \models_x \Box\tau_{\boxed{\top}}(B)$. Thus it follows that $\mathcal{N} \models_x \Box\top \to \Box\tau_{\boxed{\top}}(B)$.

For the 'if' direction suppose that $\mathcal{N} \models_x \Box\top \to \Box\tau_{\boxed{\top}}(B)$. As $N(x)$ contains the unit we know that $\|\top\| \in N(x)$ – and thus that $\mathcal{N} \models_x \Box\top$. It follows that $\mathcal{N} \models_x \Box\tau_{\boxed{\top}}(B)$. So we know that $\|\tau_{\boxed{\top}}(B)\| \in N(x)$. By the inductive hypothesis it follows then that $\|B\| \in N(x)$ and consequently that $\mathcal{N} \models_x \Box B$ as desired. ☐

Theorem 6.2.16. *For all formulas* A *we have the following.*

$$A \in \mathbf{EN} \textit{ if and only if } \tau_{\boxed{\top}}(A) \in \mathbf{E}.$$

Proof. The 'only if' direction is given by Lemma 6.2.13. For the 'if' direction suppose that $A \notin \mathbf{EN}$. Then by Theorem 6.2.14 there is a neighbourhood model $\mathcal{N} = \langle W, N, V\rangle$ which contains the unit, and a point $x \in W$ such that $\mathcal{N} \not\models_x A$. By Proposition 6.2.15 it follows then that $\mathcal{N} \not\models_x \tau_{\boxed{\top}}(A)$ and that, as this is a neighbourhood model, that $\tau_{\boxed{\top}}(A) \notin \mathbf{E}$. ☐

This translation, called $(\cdot)^*$ in Segerberg [1971a, p.212], is there used to show that for all formulas A and B that:

$$A \in \mathbf{K} + \{\Diamond\top \to B\} \text{ if and only if } \tau_{\boxed{\top}}(A) \in \mathbf{EMCD} + \{\Box\top \to \tau_{\boxed{\top}}(B)\}.$$

We can improve upon this result, which is one half of Lemma 3.4 in Segerberg [1971a, p.215], using the logical machinery introduced above. What we will now show is that we can faithfully embed **EMN** into **EM** and **EMNC** (aka. **K**) into **EMC**. First we will need the following results from Chellas [1980]. Let us say that a neighbourhood set $N(x)$ in a neighbourhood frame $\langle W, N\rangle$ is *closed under intersections* if it fulfils the following condition for all points $x \in W$.

$$(c) \quad \text{if } X \in N(x) \text{ and } Y \in N(x) \text{ then } X \cap Y \in N(x).$$

Theorem 6.2.17 (Chellas [1980] Theorem 9.12). **EMC** *is determined by the class of all supplemented neighbourhood frames whose neighbourhood sets are closed under intersections.*

Theorem 6.2.18 (Chellas [1980] Theorem 9.14). **K** *is determined by the class of all neighbourhood frames whose neighbourhood sets are supplemented, closed under intersections and contain the unit.*

Theorem 6.2.19. *For all formulas A we have the following:*

$$A \in \mathbf{EMN} \textit{ if and only if } \tau_{\boxed{\top}}(A) \in \mathbf{EM}.$$

Proof. The 'only if' direction follows from Lemma 6.2.13, with the cases of **M** in the basis case being handled by the following proof of $\tau_{\boxed{\top}}(\mathbf{M})$.

(1)	$\Box(A \wedge B) \to (\Box A \wedge \Box B)$	**M**
(2)	$\Box\top \to \Box(A \wedge B) \to \Box\top \to (\Box A \wedge \Box B)$	(1), TF
(3)	$(\Box\top \to (\Box A \wedge \Box B)) \to ((\Box\top \to \Box A) \wedge (\Box\top \to \Box B))$	TF
(4)	$\Box\top \to \Box(A \wedge B) \to ((\Box\top \to \Box A) \wedge (\Box\top \to \Box B))$	(2), (3)TF

For the 'if' direction suppose that $A \notin \mathbf{K}$. Then we know that there is a neighbourhood model $\mathcal{N}$ which is supplemented and contains the unit, and a point $x \in W$ such that $\mathcal{N} \not\models_x A$. By Proposition 6.2.15 it follows then that $\mathcal{N} \not\models_x \tau_{\boxdot}(A)$. As this is a neighbourhood model is supplemented it follows that $\tau_{\boxdot}(A) \notin \mathbf{EM}$. □

Theorem 6.2.20. *For all formulas A we have the following:*

$$A \in \mathbf{K} \text{ if and only if } \tau_{\boxdot}(A) \in \mathbf{ECM}.$$

Proof. The 'only if' direction follows from Lemma 6.2.13, with the case of **M** in the inductive step being handled as in Theorem 6.2.19 and the case of **C** being handled by the following proof of $\tau_{\boxdot}(\mathbf{C})$ in **EC**.

(1)	$(\Box A \wedge \Box B) \to \Box(A \wedge B)$	**C**
(2)	$(\Box\top \to (\Box A \wedge \Box B) \to (\Box\top \to \Box(A \wedge B))$	$(1), TF$
(3)	$(\Box\top \to (\Box A \wedge \Box B)) \leftrightarrow ((\Box\top \to \Box A \wedge \Box\top \to \Box B))$	TF
(4)	$((\Box\top \to \Box A) \wedge (\Box\top \to \Box B)) \to (\Box\top \to \Box(A \wedge B))$	$(2), (3) TF.$

For the 'if' direction suppose that $A \notin \mathbf{K}$. Then by Theorem 6.2.18 we know that there is a supplemented neighbourhood model $\mathcal{N}$ which is closed under intersections and contains the unit, and a point $x \in W$ such that $\mathcal{N} \not\models_x A$. By Proposition 6.2.15 it follows then that $\mathcal{N} \not\models_x \tau_{\boxdot}(A)$. As this is a neighbourhood model is supplemented and closed under intersections it follows from Theorem 6.2.17 then that $\tau_{\boxdot}(A) \notin \mathbf{ECM}$. □

In fact, we are able to provide a general result from which the above two results follow.

Theorem 6.2.21. *Suppose that* S *is a congruential modal logic such that* $\mathsf{S}(\boxed{\top}$
$) \supseteq \mathsf{S}$. *Then* $\mathsf{S}(\boxed{\top}) = \mathsf{S} +_e \mathbf{N}$.

Proof. That $\mathsf{S}(\boxed{\top}) \supseteq \mathsf{S} +_e \mathbf{N}$ is obvious. To see that $\mathsf{S}(\boxed{\top}) \subseteq \mathsf{S} +_e \mathbf{N}$ first note that $A \leftrightarrow \tau_{\boxdot}(A) \in \mathsf{S} +_e \mathbf{N}$. So if $A \in \mathsf{S}(\boxed{\top})$ then $\tau_{\boxdot}(A) \in \mathsf{S}$. Consequently $\tau_{\boxdot}(A) \in \mathsf{S} +_e \mathbf{N}$ and so, finally, $A \in \mathsf{S} +_e \mathbf{N}$. □

6.3 A Simple Embedding of E in monomodal K

As we saw in §6.2.1 the translational embeddings of **E** into monomodal **K** which we can see in the literature all suffer from the translated formulas having a large modal degree – the modal degree of $\tau(A)$ being 6 times the modal degree of A where $\tau(A) = \mathsf{T}_2((A)^{F'})$, and 7 times the modal degree of A where $\tau(A) = \mathsf{T}_3((A)^{F})$. What we will now give is our simplified embedding – simplified both in the sense of being direct, not taking a detour through multi-modal logic, and also in the sense that the modal degree of translated formulas is smaller than those given in the literature surveyed in §6.2.1.[16]

Let $\tau_{\Box'}$ be the modal translation which uniformly replaces all occurrences of $\Box B$ with $\Box'\tau_{\Box'}(B)$, where $\Box'$ is defined as follows.[17]

$$\Box' A =_{Def} \Diamond(\Diamond(\Box A \wedge \Box\Box\Diamond\top) \wedge \Diamond(\Diamond\Box\neg A \wedge \Diamond\Diamond\Box\bot))$$

Like every context, the context $C(p) = \Box' p$ is congruential in **K**, allowing us to conclude the following.

Lemma 6.3.1. *For all formulas A if $A \in \mathbf{E}$ then $\tau_{\Box'}(A) \in \mathbf{K}$.*

All that remains to be shown then is that this translation is faithful – which we will undertake to show model-theoretically.

Given a neighbourhood model $\mathcal{N} = \langle W, N, V\rangle$, a point $x \in W$ and a neighbourhood $X \in N(x)$ let $\langle x, X, i\rangle$ $(0 \leq i \leq 5)$ be new points not belonging to W, which we will write as $\langle x, X\rangle^*$, $\langle x, X\rangle^+$, $(x, X)^-$, $\langle x, X\rangle^I$,$\langle x, X\rangle^I_{e1}$, $\langle x, X\rangle^I_{e2}$. Here we are thinking of the labels $\langle x, X\rangle^+$ and $\langle x, X\rangle^-$ as denoting the neighbourhood $\langle x, X\rangle$ and its complement respectively. The superscript I should be read as 'intermediary', and the subscripted e in $e1$ and

[16] This material has appeared as French [2009]

[17] The first conjunct of the conjunction in the scope of the main '$\Diamond$' here could more simply be written as $\Diamond\Box(A \wedge \Box\Diamond\top)$, but the formulation above has the advantage of displaying the second conjuncts of the two inner conjunctions as being each other's negations.

$e2$ as 'dead-end'. Our reason for the choice of these names should become clear in what follows. Let $K_{\langle x,X\rangle}$ and $R_{\langle x,X\rangle}$ be defined as follows.

$$
\begin{aligned}
K_{\langle x,X\rangle} &= \{\langle x,X\rangle^*, \langle x,X\rangle^+, \langle x,X\rangle^i, (x,X)^I_{e1}, \langle x,X\rangle^I_{e2}, \langle x,X\rangle^-\}.\\
R_{\langle x,X\rangle} &= \{\langle\langle x,X\rangle^*, \langle x,X\rangle^+\rangle, \langle (x,X)^*, \langle x,X\rangle^I\rangle, \langle\langle x,X\rangle^I, \langle x,X\rangle^-\rangle,\\
&\quad \langle\langle x,X\rangle^I, \langle x,X\rangle^I_{e1}\rangle, \langle (x,X)^I_{e1}, \langle x,X\rangle^I_{e2}\rangle\}.
\end{aligned}
$$

Definition 6.3.2. Given a neighbourhood model $\mathcal{N} = \langle W, N, V\rangle$ construct the Kripke model $\mathcal{N}_{EK} = \langle W_{EK}, R_{EK}, V_{EK}\rangle$ as follows.

- $W_{EK} := W \cup (\bigcup_{x\in W}\{K_{\langle x,X\rangle} | X \in N(x)\})$.
- $R_{EK} := (\bigcup_{x\in W}\{R_{\langle x,X\rangle} | X \in N(x)\}) \cup \{\langle x, \langle x,X\rangle^*\rangle | X \in N(x)\} \cup$
  $\{\langle\langle x,X\rangle^+, y\rangle | y \in X\} \cup \{\langle\langle x,X\rangle^-, y\rangle | y \in (W \setminus X\rangle\}$.
- $V_{EK} := V$.

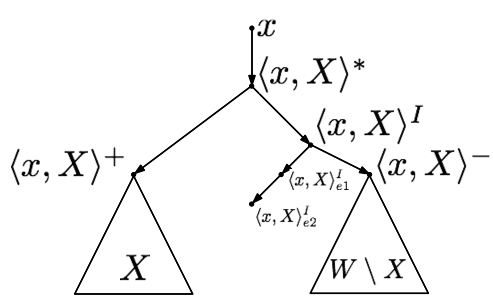

Figure 6.1: A snapshot of $\mathcal{N}_{EK}$ for a neighbourhood $X \in N(x)$.

The debt to the model constructions given in Brown [1988] and Gasquet & Herzig [1996] should be fairly clear here. To get a better idea of what this model construction does it is worthwhile looking at Figure 1, which illustrates what happens to a point $x \in W$ which has X as one of its neighbourhoods. The model construction creates a structure like that in Figure 1 for each neighbourhood X in $N(x)$ for all points $x \in W$ – the points $\langle x, X\rangle^*$ acting as 'overseers' allowing us to check whether some formula B is true throughout some neighbourhood X and false throughout its complement $W \smallsetminus X$.

One might wonder about the purpose of the 'dead end' points in the above model construction. What we will now show is that these dead end points, along with the pure formulas in the definition of $\Box' A$, allow us to force the two conjunctions within the scope of the outermost diamond to be true at specific points in the model ($\langle x, X\rangle^+$ and $\langle x, X\rangle^I$ respectively) whenever $\Box' A$ is true at $\langle x, X\rangle^*$.

Lemma 6.3.3. *For all formulas A and all $x \in W$ and $X \in N(x)$ we have the following.*

$$\mathcal{N}_{EK} \models_{\langle x,X\rangle^*} \Diamond(\Diamond\Box\neg A \wedge \Diamond\Diamond\Box\bot) \Leftrightarrow \mathcal{N}_{EK} \models_{\langle x,X\rangle^I} \Diamond\Box\neg A \wedge \Diamond\Diamond\Box\bot$$

Proof. For the '$\Rightarrow$' direction suppose that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^*} \Diamond(\Diamond\Box\neg A \wedge \Diamond\Diamond\Box\bot)$, and suppose for a reductio that $\mathcal{N}_{EK} \not\models_{\langle x,X\rangle^I} \Diamond\Box\neg A \wedge \Diamond\Diamond\Box\bot$. As the only other point R_{EK}-accessible to $\langle x, X\rangle^*$ is $\langle x, X\rangle^+$ this means that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^+} \Diamond\Box\neg A \wedge \Diamond\Diamond\Box\bot$. In particular it follows that that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^+} \Diamond\Diamond\Box\bot$. So there is a point $y \in R_{EK}(\langle x, X\rangle^+)$ such that $\mathcal{N}_{EK} \models_y \Diamond\Box\bot$. So for some z such that $R_{EK}yz$, $\mathcal{N}_{EK} \models_z \Box\bot$. But this is impossible, since the only z for which $R_{EK}yz$ is $\langle y, Y\rangle^*$, and $\langle y, Y\rangle^*$ is not a point lacking R_{EK}-successors (in fact, having precisely two, namely $\langle y, Y\rangle^+$ and $\langle y, Y\rangle^I$). So by reductio it follows that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^I} \Diamond\Box\neg A \wedge \Diamond\Diamond\Box\bot$.

The '$\Leftarrow$' direction follows trivially by the definition of truth and the construction of $\mathcal{N}_{EK}$. $\Box$

Lemma 6.3.4. *For all formulas A and all $x \in W$ and $X \in N(x)$ we have the following.*

$$\mathcal{N}_{EK} \models_{\langle x,X\rangle^*} \Diamond(\Box A \wedge \Box\Box\Diamond\top) \Leftrightarrow \mathcal{N}_{EK} \models_{\langle x,X\rangle^+} \Box A \wedge \Box\Box\Diamond\top.$$

Proof. For the '$\Rightarrow$' direction suppose that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^*} \Diamond(\Box A \wedge \Box\Box\Diamond\top)$, and suppose for a reductio that $\mathcal{N}_{EK} \not\models_{\langle x,X\rangle^+} \Box A \wedge \Box\Box\Diamond\top$. Then, as the only other point R_{EK}-accessible to $\langle x, X\rangle^{*}$ is $\langle x, X\rangle^{I}$ we know that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^{I}} \Box A \wedge \Box\Box\Diamond\top$. In particular this means that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^{I}} \Box\Box\Diamond\top$. Thus $\mathcal{N}_{EK} \models_{\langle x,X\rangle^{I}_{e1}} \Box\Diamond\top$ and $\mathcal{N}_{EK} \models_{\langle x,X\rangle^{I}_{e2}} \Diamond\top$, which cannot happen. Thus it follows that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^{+}} \Box A \wedge \Box\Box\Diamond\top$ as desired.

The '$\Leftarrow$' direction follows trivially by the definition of truth and the construction of $\mathcal{N}_{EK}$.

$\Box$

Lemma 6.3.5. *For all points $x \in W$ and $X \in N(x)$ we have the following.*

$$\mathcal{N}_{EK} \models_{\langle x,X\rangle^{+}} \Box\Box\Diamond\top.$$

Proof. Suppose for a reductio that $\mathcal{N}_{EK} \not\models_{\langle x,X\rangle^{+}} \Box\Box\Diamond\top$. Then there must be a point $y \in R_{EK}(\langle x, X\rangle^{+})$ and a point $z \in R_{EK}(y)$ such that $\mathcal{N}_{EK} \not\models_{z} \Diamond\top$. It is easy to see that such a point z must be of the form $\langle y, Y\rangle^{*}$ for some $Y \in N(y)$ and that such a point has exactly 2 R_{EK}-successors – namely $\langle y, Y\rangle^{+}$ and $\langle y, Y\rangle^{I}$ – and thus $\mathcal{N}_{EK} \models_{z} \Diamond\top$, giving us a contradiction. Thus by reductio it follows that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^{+}} \Box\Box\Diamond\top$ as desired. $\Box$

Theorem 6.3.6. *Let $\mathcal{N} = \langle W, N, V\rangle$ and $\mathcal{N}_{EK} = \langle W_{EK}, R_{EK}, V_{EK}\rangle$ be the model given by Definition 6.3.2. Then, for all formulas A and all points $x \in W$ we have the following.*

$$\mathcal{N} \models_{x} A \textit{ if and only if } \mathcal{N}_{EK} \models_{x} \tau_{\Box'}(A).$$

Proof. By induction upon the complexity of A, the only case of interest being that in the inductive step where $A = \Box B$ for some formula B.

For the 'only if' direction suppose that $\mathcal{N} \models_{x} \Box B$. Then for $X = \|B\|$, we have that $X \in N(x)$. By the inductive hypothesis it follows that for all points $y \in X$ that $\mathcal{N}_{EK} \models_{y} \tau_{\Box'}(B)$. By the definition of R_{EK} it follows that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^{+}} \Box\tau_{\Box'}(B)$ and by Lemma 6.3.5 $\mathcal{N}_{EK} \models_{\langle x,X\rangle^{+}} \Box\Box\Diamond\top$, and consequently that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^{*}} \Diamond(\Box\tau_{\Box'}(B) \wedge \Box\Box\Diamond\top)$. As the points $y \in X$ are the only points in $\mathcal{N}$ where B is true we know that all the points $z \in W \setminus X$ are such that $\mathcal{N} \not\models_{z} B$ and so by the inductive hypothesis we know that, for all such points z $\mathcal{N}_{EK} \not\models_z \tau_{\Box'}(B)$ – and hence that $\mathcal{N}_{EK} \models_z \neg\tau_{\Box'}(B)$. By the definition of R_{EK} it follows that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^-} \Box\neg\tau_{\Box'}(B)$, and thus that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^I} \Diamond\Box\neg\tau_{\Box'}(B) \wedge \Diamond\Diamond\Box\bot$. Consequently we can see that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^*} \Diamond(\Diamond\Box\neg\tau_{\Box'}(B) \wedge \Diamond\Diamond\Box\bot)$ and thus that $\mathcal{N}_{EK} \models_x \tau_{\Box'}(\Box B)$.

For the 'if' direction suppose that $\mathcal{N}_{EK} \models_x \Diamond(\Diamond(\Box\tau_{\Box'}(B) \wedge \Box\Box\Diamond\top) \wedge \Diamond(\Diamond\Box\neg\tau_{\Box'}(B) \wedge \Diamond\Diamond\Box\bot))$. Then by the definition of truth we have that there exists a point $y \in R_{EK}(x)$ such that $\mathcal{N}_{EK} \models_y \Diamond(\Box\tau_{\Box'}(B) \wedge \Box\Box\Diamond\top) \wedge \Diamond(\Diamond\Box\neg\tau_{\Box'}(B) \wedge \Diamond\Diamond\Box\bot)$. From the construction of $\mathcal{N}_{EK}$ we know that such a y will be $\langle x, X\rangle^*$ for some $X \in N(x)$. By Proposition 6.3.4 it follows that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^+} \Box\tau_{\Box'}(B) \wedge \Box\Box\Diamond\top$. Thus for all points $y \in X$ we have $\mathcal{N}_{EK} \models_y \tau_{\Box'}(B)$. By the inductive hypothesis $\mathcal{N} \models_y B$ for all such points y. By Proposition 6.3.3 it follows that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^I} \Diamond\Box\neg\tau_{\Box'}(B) \wedge \Diamond\Diamond\Box\bot$, and thus that $\mathcal{N}_{EK} \models_{\langle x,X\rangle^-} \Box\neg\tau_{\Box'}(B)$. Thus for all points $z \in W \setminus X$ we have that $\mathcal{N}_{EK} \not\models_z \tau_{\Box'}(B)$. By the inductive hypothesis $\mathcal{N} \not\models_z B$ for all such points z, from which it follows that $X = \|B\|$ and thus that $\mathcal{N} \models_x \Box B$ as desired. □

Theorem 6.3.7. *For all formulas A we have the following.*

$$A \in \mathbf{E} \textit{ if and only if } \tau_{\Box'}(A) \in \mathbf{K}.$$

Proof. The 'only if' direction is Lemma 6.3.1. For the 'if' direction suppose that $A \notin \mathbf{E}$. Then there is a neighbourhood model $\mathcal{N} = \langle W, N, V\rangle$ and a point $x \in W$ such that $\mathcal{N} \not\models_x A$. By Theorem 6.3.6 it follows that $\mathcal{N}_{EK} \not\models_x \tau_{\Box'}(A)$ and thus, as this is a model on a Kripke frame, that $\tau_{\Box'}(A) \notin \mathbf{K}$ as desired. □

So we have shown that our translation $\tau_{\Box'}$ faithfully embeds **E** into monomodal **K**. Furthermore, as promised, our translation is simpler than those which we surveyed above in section §6.2.2 – the $\tau_{\Box'}$-translation of a formula of modal degree n having a modal degree of $5n$, as compared to the $6n$ and $7n$ we get from the Thomason-derived translations.

All of the translations of **E** into monomodal **K** also allow us to bring out an interesting feature of unary contexts in **K**. One might have thought that unary contexts in **K** are in some sense special – sharing some property which does not follow from them all being congruential. What these embedding results show, though, is that there is nothing special in this sense about unary contexts in **K** – contexts like $\Box' p$ having all and only the properties of contexts that follow from their being congruential, just like the context $\Box p$ in **E**.

VII

Conclusion

In closing there are a few remarks worth making, both concerning what we have shown and some avenues for further work. What we have done here is give wide ranging and systematic study of modal-to-modal translations between monomodal logics. The focus on modal-to-modal translations was motivated from a formal perspective by the fact that modal-to-modal translations are very simple and elegant translations to work with which still allow us to cast some light on the workings of definitional translations in general. From a philosophical perspective, modal logics are still one of the best formal tools we have for investigating various live philosophical problems in analytic philosophy, and so the translations between them can provide useful formal tools for discussing the relationships between the concepts our logics are intended to characterize.

For example it can be shown that tense logic with presentist quantifiers is translationally equivalent to tense logic with eternalist quantifiers – and so these two logics are 'really the same logic' by the argument given in §5.3.[1] If we really think that our logic represents our metaphysics accurately in all the relevant respects to the dispute between presentism and eternalism then we might be moved to think that their dispute is nothing but a merely verbal one. Sider [2006] argues that this dispute is a non-verbal one by giving what he thinks is a respect in which our logics here do not accurately represent our metaphysics. Regardless of the validity or soundness of Sider's argument, what is clear here is the classificatory role which translations (and in this case translational equivalence) are playing here. Similarly, we can see the notions outlined in the preceding chapters being used implicitly (or in the case of translational equivalence, explicitly) in the arguments given in Lenzen [1979] for us regarding the normal modal logic **S4.4** as the correct epistemic logic, and **KD45** as the correct doxastic logic. The moral to be drawn here, I think, is that whenever we think we can formalize our philosophy in terms of modal logics in particular, but also in terms of logics generally, we should properly attend to how the presence of translations between the logics we are considering bears on our philosophical judgments.

Along the way we also left a variety of problems open, which we will now reiterate and comment on.

Range of Translations: We have left open a number of open problems regarding the range of translations, both very specific and very general. The main questions of interest concern the maximality of logics within the range of a translation. Throughout Chapter 4 we have a number of results which show that all of the logics fulfilling a given condition are sublogics of one of a number of maximal logics if they are in the range at all. For example, Theorem 4.2.30 shows that all Kripke-complete modal

[1]Presentist and Eternalist quantifiers are the obvious tense logical analogues of the actualist and possibilist quantifiers in Forbes [1989] – where we translate the actualist quantifier $\exists x\varphi$ as the obvious range restriction of the possibilist quantifier (i.e. $\Sigma x\varphi \wedge E(x)$), and the possibilist quantifier $\Sigma x\varphi$ in terms of the actualist quantifier and the scope shifting vlach operators $\uparrow$ and $\downarrow$ as $\uparrow \Diamond\exists x \downarrow \varphi$.

logics extending **KD4.2** which are in $NRan(\tau_{\Diamond\Box}, \mathbf{KD45})$ are sublogics of either **KD45** or **S4.4**.

The canonical example of this kind of problem, though, is the **KT**-embedding problem, outlined in Chapter 4.3 and also addressed in French & Humberstone [2009]. The **KT**-embedding problem is that of settling Conjecture 4.3.4, that all normal modal logics which **KT** can be faithfully embedded into by the $\tau_{\boxdot}$-translation all lie between **K** and **KT**. We would like to know, once and for all, whether this conjecture is true.

In a more general setting, we would like to know whether there are any other general results like Theorem 3.2.1 regarding maximality of logics in the range of a translation, and whether a set of maximal logics constitutes all of the maximal logics in the range – i.e. whether a given set of maximal logics in $NRan(\tau, \mathsf{S})$ constitute $max(NRan(\tau, \mathsf{S})$.

Intertranslatability and Translational Equivalence: There are two substantive problems concerning translational equivalence which we have left open in the text. The first of these, Open Problem 5.2.19, is whether there is a pair of normal monomodal logics which are intertranslatable but not translationally equivalent. In particular we'd like that these two logics share the same set of equivalence formulas, so that they are actually candidates for being translationally equivalent, rather than failing to be translationally equivalent by default. The second of these problems is not highlighted in the text overly much, but it relates to equivalence between logics. We would like to have a pair of consequence relations which are equipollent, but not definitionally equivalent – showing that equipollence is strictly weaker than definitional equivalence – or alternatively, prove that there can be no such pair of consequence relations, thus showing that equipollence and definitional equivalence are really just the same notion.

Open Problem 7.0.8. Is there a pair of consequence relations $\vdash_1$ and $\vdash_2$ such that $\vdash_1$ and $\vdash_2$ are equipollent, but not definitionally equivalent?

As mentioned on page 98, it would be nice to know whether Theorem

5.0.15 can be weakened to hold for logics which are merely intertranslatable.

We also have left open another interesting formal problem, Problem 5.1.13, of whether the logic $\mathbf{KDAlt}_n$ can be faithfully embedded into the logic $\mathbf{KTAlt}_{n+1}$ for $n > 1$ – the $n = 1$ case being shown in Chapter 5.1.

We have remained largely silent, except for a brief excursus in §6.2, on issues arising when we consider multi-modal logics – a good example of translations of this kind appearing in Kuhn [2004]. Given that a large amount of our philosophical theorizing usually involves considering clusters of interrelated concepts and the relationships between them (as in the example given in §2.1.1.1) translations between multi-modal logics, and the relationships such translations bear to the related monomodal logics, are well worth investigating. We will leave aside this research for another time.

Bibliography

Aanderaa, S. 1969. Relations Between Different Systems of Modal Logic. *Notices of the AMS*, **13**, 391.

Andréka, H., Madarász, J. X., & Németi, I. 2005. Mutal Definability Does not Imply Definitional Equivalence, A Simple Example. *Mathematical Logic Quaterly*, **51**, 591–597.

Blok, W. J., & Pigozzi, D. 1989. *Algebraizable Logics*. Memoirs of the American Mathematical Society Volume 77. American Mathematical Society.

Boolos, G. 1980. On Systems of Modal Logic with Provability Interpretations. *Theoria*, **46**, 7–18.

Boolos, G. 1993. *The Logic of Provability*. Cambridge University Press.

Brown, M.A. 1988. On The Logic of Ability. *Journal of Philosophical Logic*, **17**, 1–26.

Byrd, M. 1980. Eventual Permanence. *Notre Dame Journal of Formal Logic*, **21**, 591–601.

Caleiro, C., & Gonçalves, R. 2007. Equipollent Logical Systems. *Pages 97–109 of*: Beziau, J.-Y. (ed), *Logica Universalis: Towards a General Theory of Logic*, 2 edn. Basel: Birkhäuser.

Cerrito, S., & Mayer, M.C. 1997. A Polynomial Translation of **S4** into **T** and Contraction-Free Tableaux for **S4**. *Logic Journal of the IGPL*, **5**, 287–300.

Chagrov, A., & Zakharyashchev, M. 1992. Modal Companions of Intermediate Propositional Logics. *Studia Logica*, **51**, 49–82.

Chellas, B.F. 1980. *Modal Logic: An Introduction*. Cambridge: Cambridge University Press.

Corcoran, J. 1980. On Definitional Equivalence and Related Topics. *History and Philosophy of Logic*, **1**, 231–234.

Corcoran, J. 1983. Mutual Interpretability, Definitional Equivalence, and Elongations (abstract). *Journal of Symbolic Logic*, **48**, 516–517.

Crabbé, M. 1991. Deux Redondances de la Régle De Löb en Logique Modale. *Logique et Analyse*, **34**, 15–22.

Cresswell, M. J. 1967. The Interpretation of Some Lewis Systems of Modal Logic. *Australasian Journal of Philosophy*, **45**, 198–206.

Cresswell, M. J. 1988. Necessity and contingency. *Studia Logica*, **47**, 145–149.

Czelakowski, J. 1980. Equivalential logics I. *Studia Logica*, **40**, 227–236.

Dawson, E.E. 1959. A Model for Deontic Logic. *Analysis*, **19**, 73–78.

de Bouvère, K.L. 1965. Synonymous Theories. *In: The Theory of Models (Proceedings of the 1963 International Symposium at Berkeley)*. Amsterdam: North-Holland.

Demri, S., & Goré, R. 2000. An $O(n.logn)^3$-Time Transformation from Grz into Decidable Fragments of Classical First-Order Logic. *Pages 152–166 of: Selected Papers from Automated Deduction in Classical and Non-Classical Logics*. Lecture Notes in Computer Science, vol. 1761. Springer-Verlag.

Escriba, J.B. 1989. A Modal Embedding for Partial Information Semantics. *Logique et Analyse*, **125**, 131–137.

Feitosa, H.A., & D'Ottaviano, I.M.L. 2001. Conservative Translations. *Annals of Pure and Applied Logics*, **108**, 205–227.

Fitting, M. 1988. First-order Modal Tableaux. *Journal of Automated Reasoning*, **4**, 191–213.

Forbes, G. 1989. *Languages of Possibility: An Essay in Philosophical Logic*. Oxford: Basil Blackwell.

French, R. 2008. A Note on the Logic of Eventual Permanance for Linear Time. *Notre Dame Journal of Formal Logic*, **49**, 137–142.

French, R. 2009. A Simplified Embedding of **E** into Monomodal **K**. *Logic Journal of the IGPL*, **17**, 421–428.

French, R., & Humberstone, L. 2009. Partial Confirmation of a Conjecture on the Boxdot Translation in Modal Logic. *Australasian Journal of Logic*, **7**, 56–61.

Gasquet, O., & Herzig, A. 1996. From Classical to Normal Modal Logics. *Pages 293–311 of:* Wansing, H. (ed), *Proof Theory of Modal Logics*. Applied Logic Series. Kluwer Academic Publishers.

Goldblatt, R. 1993. *Mathematics of Modality*. Stanford: CSLI Publications.

Goris, E. 2007. Interpreting Knowledge into Belief in the Presence of Negative Introspection. *Technical Report TR-2007005, CUNY Ph.D. Program in Computer Science.*

Governatori, Guido, Hodkinson, Ian M., & Venema, Yde (eds). 2006. *Advances in Modal Logic 6, papers from the sixth conference on "Advances in Modal Logic," held in Noosa, Queensland, Australia, on 25-28 September 2006*. London: College Publications.

Hintikka, J. 1962. *Knowledge and Belief*. Ithaca, NY: Cornell University Press.

Hughes, G.E., & Cresswell, M.J. 1968. *An Introduction to Modal Logic*. London: Methuen.

Humberstone, L. 1993. Zero-place Operations and Functional Completeness (and the Definition of New Connectives). *History and Philosophy of Logic*, **14**, 39–66.

Humberstone, L. 1995. The Logic of Non-Contingency. *Notre Dame Journal of Formal Logic*, **36**, 214–229.

Humberstone, L. 2000. Contra-Classical Logics. *Australasian Journal of Philosophy*, **78**, 437–474.

Humberstone, L. 2004. Two-Dimensional Adventures. *Philosophical Studies*, **118**, 17–65.

Humberstone, L. 2005a. Béziau's Translation Paradox. *Theoria*, **71**, 138–181.

Humberstone, L. 2005b. Logical Discrimination. *Pages 207–228 of:* Beziau, J.-Y. (ed), *Logica Universalis: Towards a General Theory of Logic*. Basel: Birkhäuser.

Humberstone, L. 2006. Weaker-to-Stronger Translational Embeddings in Modal Logic. *In:* Governatori *et al.* [2006].

Humberstone, L. 2009. Collapsing Modalities. *Notre Dame Journal of Formal Logic*, **50**, 119–132.

Inoué, T. 1990. Unprovability-Preserving Sound Translations. *Logique et Analyse*, **33**, 243–257.

Jennings, R.E., & Schotch, P.K. 1981. Some Remarks on (Weakly) Weak Modal Logics. *Notre Dame Journal of Formal Logic*, 22, 309 – 314.

Kenny, A. 1976. Human Abilities and Dynamic Modalities. *Pages p. 209–232 of:* Manninen, J., & Tuomela, R. (eds), *Essays on Explanation and Understanding*. Dordrecht/Boston: D. Reidel Publishing Co.

Kracht, M. 1999. *Tools and Techniques in Modal Logic*. Amsterdam: Elsevier.

Kracht, M., & Wolter, F. 1997. Simulation and Transfer Results in Modal logic – A Survey. *Studia Logica*, **59**, 149–177.

Kracht, M., & Wolter, F. 1999. Normal Monomodal Logics can Simulate all Others. *Journal of Symbolic Logic*, **64**, 99–138.

Kuhn, S. 1977. *Many-sorted Modal Logics*. Uppsala: Filosofiska Studier.

Kuhn, S. 1995. Minimal Non-contingency Logic. *Notre Dame Journal of Formal Logic*, **36**, 230–234.

Kuhn, S. 2004. A Simple Embedding of T into Double S5. *Notre Dame Journal of Formal Logic*, **45**, 13–18.

Lemmon, E.J. 1966. Algebraic Semantics for Modal Logics II. *Journal of Symbolic Logic*, **31**, 191–218.

Lenzen, W. 1978. Recent Work in Epistemic Logic. *Acta Philosophica Fennica*, **30**.

Lenzen, W. 1979. Epistemologische Betrachtungen zu [S4, S5]. *Erkenntnis*, **14**, 33–56.

Litak, T. 2007. The Non-Reflexive Counterpart of Grz. *Bulletin of the Section of Logic*, **36**, 195–208.

Makinson, D. 1971. Some Embedding Theorems in Modal Logic. *Notre Dame Journal of Formal Logic*, **12**, 252–254.

Makinson, D. 1973. A Warning About the Choice of Primitive Operators in Modal Logic. *Journal of Philosophical Logic*, **2**, 193–196.

Muravitsky, A. Y. 2006. The Embedding Theorem: Its Further Developments and Consequences. Part 1. *Notre Dame Journal of Formal Logic*, **47**, 525–540.

Pelletier, F.J. 1984. Six Problems in "Translational Equivalence". *Logique et Analyse*, **108**, 423–434.

Pelletier, F.J., & Urquhart, A. 2003. Synonymous Logics. *Journal of Philosophical Logic*, **32**, 259–285.

Pelletier, F.J., & Urquhart, A. 2008. Synonymous Logics: A Correction. *Journal of Philosophical Logic*, **37**, 95–100.

Porte, J. 1980. Congruences in Lemmon's S0.5. *Notre Dame Journal of Formal Logic*, **21**, 672–678.

Prawitz, D., & Malmnäs, P.-E. 1968. A Survey of some Connections Between Classical, Intuitionistic and Minimal Logic. *Pages 215–219 of:* H. Schmidt, K. Schütte, & Thiele, H.-J. (eds), *Contributions to mathematical logic*. Amsterdam: North-Holland.

Rawling, J.P., & Selesnick, S.A. 2000. Orthologic and Quantum Logic: Models and Computational Elements. *Journal of the ACM*, **47**, 721 – 751.

Rescher, N., & Urquhart, A. 1971. *Temporal Logic*. Vienna: Springer.

SALMON, N. 1989. The Logic of What Might Have Been. *Philosophical Review*, **98**, 3–34.

SEGERBERG, K. 1967. On the Logic of 'To-morrow'. *Theoria*, **33**, 46–52.

SEGERBERG, K. 1968. Propositional Logics Related to Heyting's and Johansson's. *Theoria*, **34**, 26–61.

SEGERBERG, K. 1971a. *An Essay in Classical Modal Logic*. Uppsala: Filosofiska Studier.

SEGERBERG, K. 1971b. On the extensions of S4.4. *Journal of Symbolic Logic*, **36**, 590–591.

SEGERBERG, K. 1982. *Classical Propositional Operators: An Exercise in the Foundations of Logic*. Oxford University Press.

SEGERBERG, K. 1986. Modal Logics with Functional Alternative Relations. *Notre Dame Journal of Formal Logic*, **27**, 504–522.

SHAVRUKOV, V. YU. 1991. On Two Extensions of the Logic of Provability GL. *Math. USSR. Sbornik*, **69**, 255–270.

SIDER, T. 2006. Quantifiers and Temporal Ontology. *Mind*, **115**, 75–97.

SILVA, J.J. DA, D'OTTAVIANO, I.M.L., & SETTE, A.M. 1999. Translations Between Logics. *Pages 435–448 of:* CAICEDO, X., & MONTENEGRO, C.H. (eds), *Models, Algebras, and Proofs*. New York: Marcel Dekker.

SMILEY, T. 1962. The Independence of Connectives. *Journal of Symbolic Logic*, **27**, 426–436.

STALNAKER, R. 2006. On Logics of Knowledge and Belief. *Philosophical Studies*, **128**, 169–199.

THOMAS, I. 1967. A Theorem on S4.2 and S4.4. *Notre Dame Journal of Formal Logic*, **8**, 335–336.

Thomason, S.K. 1974. Reduction of Tense Logic to Modal Logic. I. *Journal of Symbolic Logic*, **39**, 549–541.

Thomason, S.K. 1976. Reduction of Tense Logic to Modal Logic. II. *Theoria*, **12**, 154–169.

Tokarz, M., & Wójcicki, R. 1971. The Problem of Reconstructability of Propositional Calculi. *Studia Logica*, **28**, 119–127.

van Benthem, J. 1983. *Modal Logic and Classical Logic*. Bibliopolis.

van Benthem, J. 1986. Partiality and Nonmonotonicity in Classical Logic. *Logique et Analyse*, **114**, 225–247.

van Heijenoort, J. 1967. Harvard University Press.

Veltman, F. 1981. Data Semantics. *In:* et. al., J. Groenendijk (ed), *Formal Methods in the Study of Language*. Mathematical Center.

Williamson, T. 1993. Some Admissible Rules in Nonnormal Modal Systems. *Notre Dame Journal of Formal Logic*, **34**, 378–400.

Williamson, T. 1996. Some Admissible Rules in Modal Systems with the Brouwerian Axiom. *Logic Journal of the IGPL*, **4**, 283–303.

Williamson, T. 1998. Iterated Attitudes. *Proceedings of the British Academy*, **95**, 85–133.

Wójcicki, R. 1984. *Lectures on Propositional Calculi*. Ossolineum.

Wójcicki, R. 1988. *Theory of Logical Calculi*. Dordrecht: Kluwer.

Zakharyaschev, M., & Chagrov, A. 1997. *Modal Logic*. Oxford: Oxford University Press.

Zolin, E.E. 2000. Embeddings of Propositional Monomodal Logics. *Logic Journal of the IGPL*, **8**, 861–882.

Candidate's Publications

7.1 Papers Published

French, R. 2008. A Note on the Logic of Eventual Permanence for Linear Time. *Notre Dame Journal of Formal Logic*, **49**(2), 137–142.

French, R. 2009. A Simplified Embedding of **E** into Monomodal **K**. *Logic Journal of the IGPL*, **17**, 421–428.

French, R., & Humberstone, L. 2009. Partial Confirmation of a Conjecture on the Boxdot Translation in Modal Logic. *Australasian Journal of Logic*, **7**, 56–61.

Index