Proofs and Models in Naive Property Theory: A Response to Hartry Field's "Properties, Propositions and Conditionals"

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Abstract

In our response Field's "Properties, Propositions and Conditionals", we explore the methodology of Field's program. We begin by contrasting it with a proof-theoretic approach and then commenting on some of the particular choices made in the development of Field's theory. Then, we look at issues of property identity in connection with different notions of equivalence. We close with some comments relating our discussion to Field's response to Restall (2010).

Hartry Field's article, "Properties, Propositions and Conditionals"[1] presents an overview of a program of exploring naive theories of truth and properties in non-classical logics. Field presents many different options for developing a theory of truth and properties. This gives the reader a good sense of the current state of Field's program as well as directions for future work. The paper is rich, and while there are many themes we could address in this paper, our focus is on the methodology of Field's program. We aim to situate Field's approach to the semantic paradoxes alongside some other approaches, to thereby shed some light on some of the features of Field's particular account.

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[1]Hereafter, we will abbreviate 'Properties, Propositions and Conditionals' by 'PPC'.

We will begin by sketching out one path not taken by Field, namely that of proof-theoretic semantics. We will argue that proof-theoretic semantics offers some natural options for the sort of logic that Field is after. This leads to our discussion of certain choices Field makes with respect to his own approach, looking at motivations for other routes that were not adopted. With the significance of Field's approach thereby clarified, we then turn to Field's model construction, and discuss their connection to property identity. Finally, we will close with some issues connected to logical consequence, extensionality, and the response by Field (2010) to an objection of Restall (2010).

To start, suppose we have, in our language, some device for specifying *properties*. We supplement the formal language of first order logic with the device of "lambda abstraction." For each open sentence $\phi(x)$ with the variable x free, we have a singular term $\lambda x.\phi(x)$, which we intend to interpret as denoting the *property* of *being* ϕ. Given such a language, the question arises: when does a given object s bear the property $\lambda x.\phi(x)$. A plausible immediate response is when and only when $\phi(s)$, when s satisfies the condition we used to specify the property. This motivates two inference rules governing the instantiation relation ξ:

$$\frac{\phi(s)}{s\ \xi\ \lambda x.\phi(x)}\ \xi I \qquad \frac{s\ \xi\ \lambda x.\phi(x)}{\phi(s)}\ \xi E$$

If $\phi(s)$ holds, we can infer that s has the property of being a ϕ, and conversely, when s has that property, we can infer $\phi(s)$. So much seems straightforward. However, things cannot be quite this simple: once have inference rules like ξI and ξE, then paradox threatens. Consider the property h of being *heterological*, that is, being *non-self-instantiating*: $\lambda x.(x \not\xi\ x)$. Using our two rules ξI and ξE, it seems that we can prove $h\ \xi\ h$, and at the same time, prove $h \not\xi\ h$, a contradiction. The core steps of this derivation are ξI and ξE when applied to $h\ \xi\ h$, that is, to $\lambda x.(x \not\xi\ x)\ \xi\ \lambda x.(x \not\xi\ x)$.

$$\frac{\lambda x.(x \not\xi\ x) \not\xi\ \lambda x.(x \not\xi\ x)}{\lambda x.(x \not\xi\ x)\ \xi\ \lambda x.(x \not\xi\ x)}\ \xi I \qquad \frac{\lambda x.(x \not\xi\ x)\ \xi\ \lambda x.(x \not\xi\ x)}{\lambda x.(x \not\xi\ x) \not\xi\ \lambda x.(x \not\xi\ x)}\ \xi E$$

Combining these steps with standard inference principles for negation (and abbreviating '$\lambda x.(x \not\xi\ x)\ \xi\ \lambda x.(x \not\xi\ x)$' as '$h\ \xi\ h$' to save space), we can derive an arbitrary conclusion, like so:

$$
\begin{prooftree}
\AxiomC{$[h \mathrel{\xi} h]^1$}
\RightLabel{ξE}
\UnaryInfC{$h \mathrel{\not\xi} h$}
\AxiomC{$[h \mathrel{\xi} h]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$h \mathrel{\not\xi} h$}
\AxiomC{$[h \mathrel{\xi} h]^2$}
\RightLabel{ξE}
\UnaryInfC{$h \mathrel{\not\xi} h$}
\AxiomC{$[h \mathrel{\xi} h]^2$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$h \mathrel{\not\xi} h$}
\RightLabel{ξI}
\UnaryInfC{$h \mathrel{\xi} h$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{P}
\end{prooftree}
$$

Any response to the paradoxes must find fault with this derivation somewhere, with the steps ξI, ξE, $\neg I$, $\neg E$, or $\bot E$. Or at least, you must do so if you wish to avoid being able to prove *everything*. Of course, finding fault with one derivation is easy: simply pick a principle used in the derivation and blame *it*.[2] However, the heterological paradox is by no means alone in causing trouble for naive theories of properties. Curry's paradox arises when we consider the property $\lambda x.(x \mathrel{\xi} x \to P)$, which we will abbreviate as 'c' to save space:

$$
\begin{prooftree}
\AxiomC{$[c \mathrel{\xi} c]^1$}
\RightLabel{ξE}
\UnaryInfC{$c \mathrel{\xi} c \to P$}
\AxiomC{$[c \mathrel{\xi} c]^1$}
\RightLabel{$\to E$}
\BinaryInfC{P}
\RightLabel{$\to I^1$}
\UnaryInfC{$c \mathrel{\xi} c \to P$}
\AxiomC{$[c \mathrel{\xi} c]^2$}
\RightLabel{ξE}
\UnaryInfC{$c \mathrel{\xi} c \to P$}
\AxiomC{$[c \mathrel{\xi} c]^2$}
\RightLabel{$\to E$}
\BinaryInfC{P}
\RightLabel{$\to I^2$}
\UnaryInfC{$c \mathrel{\xi} c \to P$}
\RightLabel{ξI}
\UnaryInfC{$c \mathrel{\xi} c$}
\RightLabel{$\to E$}
\BinaryInfC{P}
\end{prooftree}
$$

In this proof of the arbitrary conclusion P, the negation rules $\neg I$, $\neg E$ and $\bot E$, are not used, but the rules $\to I$ and $\to E$ for the conditional are. To block this derivation, blame must be placed elsewhere. The only inference rules shared between the two derivations are ξI and ξE, so the natural place to start is with *these* rules, and to reject the naive theory of properties. That is not Field's approach, and his

[2] Friends of truth-value gaps blame $\neg I$, while friends of truth-value gluts either blame $\neg E$ (if $\bot$ is taken to be an automatically trivialising proposition, entailing everything) or $\bot E$ (if $\bot$ is taken to be a contradiction, not necessarily entailing everything). Another possible point of blame is not any individual inference step but the way that they are put together—the transitivity of proofs, or the *Cut* rule. For more on non-transitive responses to the paradoxes, see Cobreros et al. (2013) and Ripley (2012b,a, 2013). One can also reject the starting points of proofs—the reflexivity of proofs, for which see French (2016).

project, described in PPC, and in greater length elsewhere (Field, 2008) is to construct models that allow us to vindicate not only ξI and ξE in their full generality, but also as many of the standard logical principles concerning the standard connectives as possible.[3]

In the next section, we will introduce one systematic way to locate a fault in these paradoxical derivations and to reassure us that ξI and ξE can be safe, and compatible with a wide range of logical principles, without having to swallow the conclusions of paradoxical derivations like the two we have seen. This approach, which will directly use the tools of proof-theoretic semantics, will provide a helpful contrast to Field's account.

1 Proof-theoretic semantics

One way to assess the inference rules used in our paradoxical derivations is use criteria indigenous to the theory of proofs. One criterion is the condition of *harmony*. The fundamental inference rules for a concept (like those for negation, or the conditional, or property instantiation, which we have seen) split into *introduction* and *elimination* rules. If the introduction and elimination rules for a concept are in an appropriate kind of balance – so-called *harmony*[4] – we can draw significant conclusions about proofs constructed from those rules. In the case of the rules for the conditional, or negation, this means that a proof in which the concept is introduced, and then eliminated, can be *simplified* into one in which that detour does not take place.

$$\dfrac{\dfrac{\begin{array}{c}[A]^i\\ \vdots\,\pi_1\\ \bot\end{array}}{\neg A}\,\neg I^i \qquad \begin{array}{c}\vdots\,\pi_2\\ A\end{array}}{\bot}\,\neg E \quad\rightsquigarrow\quad \begin{array}{c}\vdots\,\pi_2\\ A\\ \vdots\,\pi_1\\ \bot\end{array} \qquad\qquad \dfrac{\dfrac{\begin{array}{c}[A]^j\\ \vdots\,\pi_3\\ B\end{array}}{A \to B}\,{\to}I^j \qquad \begin{array}{c}\vdots\,\pi_4\\ A\end{array}}{B}\,{\to}E \quad\rightsquigarrow\quad \begin{array}{c}\vdots\,\pi_4\\ A\\ \vdots\,\pi_3\\ B\end{array}$$

Here, the reductions simplify a proof by eliminating a local maximum in complexity. Instead of going through the complex formula $\neg A$, in the reduced proof,

[3]Throughout we will restrict our attention to the logical vocabulary $\{\wedge, \vee, \neg, \to, \lambda, \xi, \forall\}$.

[4]The literature on proof-theoretic harmony is extensive. Prawitz's (1965) and Dummett's (1991) are the classic texts. Steinberger (2011) is a good contemporary account of different ways to make the notion of harmony precise.

π_1 and π_2 are joined by at simpler formula, A. For the conditional reduction, instead of going through the complex formula $A \to B$, in the reduced proof, π_3 and π_4 are joined at the simpler formula, A. If every inference rule is in harmony like this, the process of normalisation eliminates such *I*/*E* detours, and produces a proof in which there are no detours through local maxima, and every formula in the proof is a subformula of a premise or conclusion of the proof (Prawitz, 1965). Proofs which satisfy this subformula property are *analytic* in a strong sense: the connections between the premises and the conclusion are given purely by inference rules governing the constituents of those formulas. There is no need to go outside those formulas into other vocabulary.

If you look at the ξI and ξE rules, they seem to be just as harmonious as the rules for the connectives. After all, any detour through a pair of ξI and ξE steps can be reduced as follows:

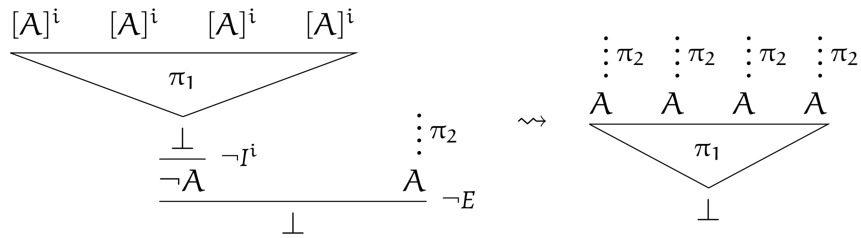
$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_5\\ \phi(s)\end{array}}{s\ \xi\ \lambda x.\phi(x)}\,\xi I}{\phi(s)}\,\xi E \quad\rightsquigarrow\quad \begin{array}{c}\vdots\,\pi_5\\ \phi(s)\end{array}$$

This reduction simplifies a proof in a way similar to the reductions for the connectives, but it differs in one very important respect. The intermediate formula cut out of the derivation (here, $s\ \xi\ \lambda x.\phi(x)$) need *not* be more complex than the formulas surrounding it. In the case where $s\ \xi\ \lambda x.\phi(x)$ is $h\ \xi\ h$ (the heterological property is self-instantiated) or $c\ \xi\ c$ (the Curry property is self-instantiated), the formula is inferred from $h \not\xi\ h$, or from $c\ \xi\ c \to P$, both of which are *more* complex than the introduced formula. Reducing the proof by snipping out the *I*/*E* pair does not involve cutting out a local maximum in formula complexity. In this case, the reduction simplifies the proof, not by making the formulas involved less complex, but by making the proof strictly *smaller*.

The property rules, then, are well behaved from a proof-theoretic perspective on one measure: normalising property rule detours invariably shrinks a proof.[5] It might seem that all normalisation steps shorten proofs, since normalising always involves eliminating detour steps. However, the reductions for the negation rules

[5]That is, the number of formulas in the proof is reduced by two. It may also reduce in *height* (the length of the longest branch) if the reduction occurs in that branch. Whenever we talk about the *size* of a proof, we mean the number of formula occurrences in the proof tree.

or for the conditional rules do *not* always result in smaller proofs.[6] If the formula discharged in a $\neg I$ or $\to I$ step is discharged more than once, then the proof of that formula is substituted into the reduced proof a number of times, like this:



If the substituted proof π_2 is comparatively large, the result of this reduction may be significantly larger than the original, non-reduced proof. Reduction, here, is reduction in complexity, not reduction in *size*.

So, we have a mismatch between the transformations that reduction steps for connectives (on the one hand), and property rules (on the other) work on proofs. Reductions for connectives reduce complexity while allowing proofs to grow in size. Reductions for the property rules do not reduce complexity, but they do always shrink proofs.[7]

This suggests a way to restore balance to our reductions. If we aim to keep ξI and ξE, we can retain the fruits of harmonious rules by demanding that all reductions shrink proofs. If we restrict $\neg I$ and $\to I$ by allowing at most one formula to be discharged at a time, then the reduction steps for the connectives reduce size as well as complexity. Once all reductions shrink proofs, and any non-normal proof can be systematically converted into a normal proof by applying the reduction

[6]You can see this in action by applying the reductions to the paradoxical proofs we have seen. The proof of P from the heterological paradox is not normal in that a $\neg I$ step is followed by a $\neg E$ step. Reducing this gives you another proof, this time, not normal because a ξI step is followed by a ξE step. Reducing *this* returns you to the original proof. This shows that the two reductions are 'reducing' along different measures. One reduces size but not complexity. The other reduces complexity but not size. Hence, the two kinds of reduction do not result in a simplification of the original proof, but in a tussle, between two kinds of simplicity.

[7]This observation is not original with us. For example, in the context of proof rules in a sequent calculus for a logic for Kripke's fixed point construction, Michael Kremer writes "the T-introduction … rules may often decrease logical complexity: no matter how complex A is, T'A' is atomic. Nor will it help to introduce a new notion of complexity, counting sentences with more occurrences of T as more complex, say. For then the quantifier rules may decrease complexity. For example, T'∃xTx' is an instance of ∃xTx, yet the former would be more complex than the latter on the proposed revised definition of complexity. In short, for the purposes of a proof-theoretic argument for the cut- elimination theorem, we would want to count T'∃xTx' as both more and less complex than ∃xTx — but we can't have it both ways" (Kremer, 1988, p. 260).

steps, one by one. Normal proofs satisfy the subformula property, and this has immediate and powerful consequences. There can be no paradoxical derivations of arbitrary conclusions using these rules. (Here is why: there is no normal proof of an atomic formula p from no premises. Since p has no subformulas at all, no introduction or elimination rules could feature in any such proof satisfying the subformula property.) So, the addition of our powerful property rules cannot interact with our logical vocabulary in this devastating way, if our logical rules are restricted in the way we have sketched.

Natural considerations arising out of proof-theoretic semantics motivate a principled way to retain strong property rules, consistent with restrictions on logical vocabulary. Noticing this is not original to us. That the rule of Contraction (allowing multiple discharge in rules like $\to I$ and $\neg I$) is implicated in the paradoxes has been known for many years. Early work by Fitch (1936; 1942) makes that connection clear.[8] Work by Grišin (1974; 1982) gives a consistency proof for a logic with rules like our $\xi I/E$ in the sequent calculus, showing that without the rule of Contraction, normalisation (cut-elimination) leads to a direct consistency proof. This work has been taken up in recent years by a number of different authors (Cantini, 2003; Girard, 1998; Petersen, 2000, 2003; Zardini, 2011, 2013).[9] That work is most often presented in the sequent calculus, rather than in a natural deduction format, but this is (for our purposes) an inessential difference.[10] While we have here focussed only on negation and the conditional, the language can be enriched with conjunction, disjunction, and quantifiers, with no difficulty at all. Provided that the rules are presented as introduction and elimination rules, and the reduction steps shrink proofs, we are assured that the paradoxes do not threaten, and the property rules are a conservative addition to the rules for the logical vocabulary.[11]

[8] See Rogerson (2007) for discussion of Fitch and Prawitz on Contraction and paradox.

[9] For more on approaches to paradox that reject the structural rule of contraction, see Shapiro (2011, 2013, 2014), Beall and Murzi (2013), Weber (2013), Caret and Weber (2014), Mares and Paoli (2014) and Shapiro and Murzi (2015), among others.

[10] This Gentzen-style natural deduction format works well, up to a point. If you wish to hew as close as possible to classical logic, and to allow $\neg\neg A$ to be equivalent to A, it is better to use a sequent system that allows for multiple formulas in conclusion position as well as multiple premises. With that small change, negation can be involutive, while keeping the restriction on discharging rules like those for negation and the conditional.

[11] To be specific, we can have both *multiplicative* and *additive* conjunction and disjunction (multiplicative disjunction in a multiple conclusion setting, at least) as well as an additive conditional, in addition to a multiplicative conditional. The standard first-order quantifiers have rules that normalise while reducing proof length. Details of the proof are straightforward (Cantini, 2003; Girard, 1998; Petersen, 2000).

So, if the aim of the exercise is to provide a principled restriction of logical vocabulary so as to keep the logical world safe for rules like ξI and ξE, then proof-theoretic considerations concerning intrinsic features of inference rules point to *Contraction* as a prime suspect. Doing away with Contraction makes our world safe for $\xi I/E$, and we can assure ourselves of this on purely proof-theoretic grounds.

2 Permutation and the rules of the game

The proof-theoretic approach just sketched provides one response to the paradoxes that preserves the naive property instantiation rules. It is not the approach Field adopts, which leads us to our main question for this part of the paper. Why not adopt the proof-theoretic approach outlined in the previous section? We are interested, in particular, in the reasons for the choice of logic for the naive theory. Given the stated commitments of PPC, some principles, such as Contraction, are out, some principles are in, such as one form of Weakening, and some appear to be up for grabs. The proof-theoretic approach can reject Contraction, accept Weakening, and provide a plausible story about the principles that are up for grabs. To sharpen the issue, we will take some of Field's comments about the rule of Permutation as our jumping off point.

Field says, "The Permutation rule lacks obviousness in a naive theory."[12] This is a comment on his "Lukasiewicz done better" semantics, which permits Permutation for a wide range of paradoxical sentences, but not all. The rule of Permutation is this inference:

$$\frac{A \to (B \to C)}{B \to (A \to C)}$$

which is valid in classical propositional logic, in intuitionistic logic, and in many other non-classical logical systems. Field's reason for resisting Permutation is that any justification for it which involves the inferences:

$$\frac{A \to (B \to C)}{(A \wedge B) \to C} \qquad \frac{(B \wedge A) \to C}{B \to (A \to C)}$$

will be found wanting, because on Field's preferred approach to the paradoxes, we must clearly distinguish a doubly nested conditionals like $A \to (A \to C)$ from $A \to C$, because of the paradoxes. But if $A \to (B \to C)$ is equivalent to $(A \wedge B) \to C$, and if $A \wedge A$ is equivalent to A, paradox threatens. This raises

[12] PPC 18

two issues, one concerning the target logic and one concerning the problem with Permutation.

While Field notes that there are different ways to define validity, he exhibits some preference for a definition that yields a paracomplete logic, that is, a logic in which the law of excluded middle is invalid. In PPC, the target logic approximates Lukasiewicz's continuum valued logic, $L_{\aleph}$. The choice of $L_{\aleph}$ for the logical vocabulary $\{\rightarrow, \wedge, \vee, \neg, \forall\}$, is in some respects unsurprising. It is a well-behaved logic with fairly simple models over the $[0, 1]$ interval. Moreover, it reduces to classical logic in certain models. In some respects, however, it is a surprising choice. It is not possible to extend the interpretation of ξ from the quantifier-free language to the language with quantifiers.[13] Since it is not possible to use $L_{\aleph}$ with quantifiers and the naive property instantiation rules, there remains a question about what which axioms and rules from $L_{\aleph}$ to keep, and which to reject. As noted in §1, there are other options for logics that share many of the good features that Field is after without the problems $L_{\aleph}$ has combining quantifiers and ξ.

There is a feature of the choice of connectives in the language that deserves comment as well. One of the usual connectives of $L_{\aleph}$ is omitted, namely fusion, $\circ$, for which the following is valid.[14]

$$((A \circ B) \rightarrow C) \leftrightarrow (A \rightarrow (B \rightarrow C))$$

While this equivalence may appear similar to the inferences displayed above, it is importantly different, as $A \circ A$ is not equivalent to A; and $\circ$ is not $\wedge$, though it is a kind of conjunction. There are non-triviality proofs for naive property theory over many logics with fusion, although not all logics that non-trivially support naive property theory do so with the addition of fusion.[15] The proof-theoretic approach of the previous section can be augmented with fusion, for which the above equivalence is derivable. For logics for which the displayed biconditional does not hold, a weaker, bidirectional rule form often does. The omission of fusion from Field's logic may be from particular philosophical concerns. Or, it may be that Field's target logic includes some axioms that lead to trouble with naive property theory when fusion is available.

To return to the question of whether the rule of Permutation lacks obviousness in the context of a logic for naive theories of properties, it depends very much

[13] See Restall (1992) and Hájek et al. (2000).
[14] See Restall (1992)
[15] See Øgaard (2015, §6).

on the angle from which one is examining the rule. Given the proof-theoretic perspective taken in §1, Permutation is independently compelling, and requires no route through conjunction to justify it at all. The proof is straightforward.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to (B \to C) \qquad [A]^1}{B \to C}{\scriptstyle \to E} \qquad [B]^2}{C}{\scriptstyle \to E}}{A \to C}{\scriptstyle \to I^1}}{B \to (A \to C)}{\scriptstyle \to I^2}$$

Permutation is plausible and, as Field notes, it is useful, so it would be good to have more or a story about why one should give it up. There are proponents of naive property theory, such as Ross Brady, who reject Permutation for systematic reasons. It will be useful to look at Brady's reasons for rejecting Permutation.

The naive property theory developed by Brady rejects Permutation for explicit philosophical reasons.[16] Brady's preferred logic and naive theories of properties and sets are presented axiomatically. The logic is motivated, not by appeal to the proof-theoretic concerns of §1, but by a view about the meaning of the conditional. He appeals to some of the standard objections to the material conditional to motivate the adoption of a *relevant logic*.[17] The particular relevant logic he opts for is motivated by appeal to meaning containment: The conditional of the logic expresses the containment of the consequent's content in that of the antecedent. We will not delve into the details of Brady's theory of meaning containment here, but it provides Brady with the resources to argue against several axioms, including Permutation and Contraction.[18] Brady also uses the theory to argue in favor of the axioms he does adopt, in particular Conjunctive Syllogism, $((A \to B) \land (B \to C)) \to (A \to C)$. Conjunctive Syllogism is a mild form of Contraction, so its inclusion in a theory of naive properties is surprising. Its inclusion is, however, motivated by Brady's theory of meaning containment.

Brady is after a non-trivial theory of naive properties and sets that respects his views concerning the conditional and meaning containment. Permutation, and other principles, are rejected on grounds stemming from those views. Field rejects Brady's views on the conditional and meaning containment, so they cannot

[16] We will focus on Brady's (2006) presentation. We will note that more recently, Brady and Meinander (2012) reject the axiom form of distribution, $(A \land (B \lor C)) \to ((A \land B) \lor (A \land C))$.

[17] Brady (2006, 2–5). The interested reader should consult Dunn and Restall (2002) and Bimbó (2006) for overviews of relevant logics.

[18] Brady (2006, 29–30)

provide a route to rejecting Permutation for Field. Permutation is not validated by the models, so there is a formal reason for rejecting Permutation. That formal reason is philosophically satisfying to the extent that the models are philosophically compelling. The models seem to play only the role of demonstrating that certain naive theories are non-trivial.[19] This leaves certain choices, such as the rejection of Permutation, feeling philosophically ill-motivated, though formally justified. We have seen why Field is not a proponent of a relevant logical response to the paradoxes, but we are not sure why he doesn't adopt a proof-theoretic approach, along the lines sketched in the previous section. We leave this question hanging, and we turn to issues of property identity, starting with some points connected to Field's model construction and then looking at his response to Restall (2010) on extensionality and triviality.

3 Models and property identity

In this section, we will look at Field's model construction using revision sequences. We will focus on the question of what the models tell us about property identity.

First some definitions to make the paper more self-contained. *Hypotheses* are functions from conditional formulas to the real interval $[0, 1]$. Formulas are evaluated relative to a model M and a hypothesis h, written $M + h$.[20] We will write $|A|_{M+h}$ for the semantic value of A in the least Kripke fixed point for instantiation over the model M and hypothesis h. The *revision operator* $\mathcal{R}^*$ maps hypotheses to hypotheses according to the following pointwise definition.

$$(\mathcal{R}^*(h))(A \to B) \begin{cases} \frac{1}{2}\big(h(A \to B) + 1\big) & \text{if } |A|_{M+h} \leq |B|_{M+h} \\ \frac{1}{2}\big(h(A \to B) + 1 \\ \qquad -(|A|_{M+h} - |B|_{M+h})\big) & \text{otherwise} \end{cases}$$

To construct the sequences, we need an initial hypothesis, h. As Field notes, there are options here, but we will focus on the hypotheses that assign $\frac{1}{2}$ to all conditionals. Let On be the class of all ordinals. A revision sequence is an On-length sequence of hypotheses such that $h_{\alpha+1} = \mathcal{R}^*(h_\alpha)$, for all ordinals α, and at limit stages, certain coherence constraints are obeyed. For purposes of this paper, we

[19] Cf. the more substantive philosophical role attributed to revision sequences by Gupta and Belnap (1993).

[20] We follow Field in using parameterized formulas, suppressing the variable assignments. We also suppress the detail about the hypotheses being transparent.

can focus on the limit rule that $h_\lambda(A \to B) = r$, if the sequence $\langle |A \to B|_{M+h_\beta} : \beta < \lambda\rangle$ converges to r and is $\frac{1}{2}$ otherwise.[21]

Formulas will exhibit different patterns of stability over the revision sequence, either stabilizing to some value or oscillating. There is a nice regularity to the sequences, as there is bound to be a stage after which hypotheses show up in cycles.[22] As Field notes, for defining validity, one can ignore the portion of the sequence before the loop starts and focus on the semi-open interval consisting of a cycle of the loop.

Let us look at the revision sequences for some simple formulas, the earlier example of $c\ \xi\ c \to \bot$, as well as $\top \to \bot$, and $(\top \to \bot) \to \bot$, which will feature in the discussion to come. In addition let c_2 abbreviate $\lambda x.(x\ \xi\ x \to (x\ \xi\ x \to \bot))$, so $c_2\ \xi\ c_2$ is equivalent to $c_2\ \xi\ c_2 \to (c_2\ \xi\ c_2 \to \bot)$. The values for these formulas in the minimal fixed point over the hypotheses are given in Table 1. These examples are fairly simple and they all stabilize from stage ω onwards. The

	0	1	2	3	4	5	$\cdots$	ω
$\top \to \bot$	1/2	1/4	1/16	1/32	1/64	1/128	$\cdots$	0
$(\top \to \bot) \to \bot$	1/2	1/2	5/8	3/4	27/32	29/32	$\cdots$	1
$c\ \xi\ c$	1/2	1/2	1/2	1/2	1/2	1/2	$\cdots$	1/2
$c\ \xi\ c \to \bot$	1/2	1/2	1/2	1/2	1/2	1/2	$\cdots$	1/2
$c_2\ \xi\ c_2$	1/2	3/4	3/4	11/16	21/32	85/128	$\cdots$	2/3
$c_2\ \xi\ c_2 \to \bot$	1/2	1/2	3/8	5/16	5/16	21/64	$\cdots$	1/3
$c_2\ \xi\ c_2 \to (c_2\ \xi\ c_2 \to \bot)$	1/2	3/4	3/4	11/16	21/32	85/128	$\cdots$	2/3

Table 1: Revision sequences

revision patterns for these examples do not change with variation of the ground model. The only thing that will affect the revision sequences is choice of initial hypothesis, and even that does not change much. For example, both $\top \to \bot$ and $(\top \to \bot) \to \bot$ are bound to stabilize to 0 and 1, respectively, from stage ω onwards.

Field raises the question of when distinct formulas can be taken to express the same property or proposition, focussing on properties for simplicity. It would be

[21] See Campbell-Moore (2019) for a general discussion of limit rules for revision sequences.

[22] This is a general feature of revision sequences, as guaranteed by the Reflection Theorem of Gupta and Belnap (1993, 172). See also Field (2016, §4).

natural for, say, $\lambda x.(p \wedge q)$ and $\lambda x.(q \wedge p)$ to be the same properties, collapsing distinct properties via some equivalence relation. Of particular interest here are the falsehoods $\lambda x.\bot$ and $\lambda x.(\top \rightarrow \bot)$. As evident from Table 1, for any revision sequence on any ground model, $\bot$ and $\top \rightarrow \bot$ both get value 0 from stage ω onwards and the biconditional between them gets the value 1 from stage ω onwards. One might think that one could stipulate $\lambda x.\bot = \lambda x.(\top \rightarrow \bot)$, but Field provides an argument, due to Øgaard, that $\lambda x.\bot$ and $\lambda x.(\top \rightarrow \bot)$ must be kept distinct, provided identity obeys some plausible conditions.[23] Field says,

> [L]aws like $(\top \rightarrow \bot) \rightarrow \bot$, though in some sense valid, don't have the kind of 'uniform validity' that is required for predicates coextensive by virtue of them to be sensibly regarded as expressing the same property.[24]

We will try to unpack this idea of uniform validity in order to better understand why we cannot take $\top \rightarrow \bot$ and $\bot$ to express the same property. To do this, it will be useful to distinguish two forms of equivalence. Two formulas A and B are *weakly equivalent* if and only if $A \models B$ and $B \models A$, that is, if every model satisfying A also satisfies B, and vice versa. The formulas are *strongly equivalent* if and only if $A \leftrightarrow B$ is valid, i.e. $\models A \leftrightarrow B$.[25] In terms of models, these say very different things. In Field's setting, the weak equivalence of A and B requires that for any revision sequence, A eventually receives only the value 1 iff B does. Strong equivalence requires that the biconditional $A \leftrightarrow B$ eventually receive only the value 1 in the revision construction, which is to say that A and B eventually receive the same values in each stage of the revision construction. To illustrate the difference, take $h \ \xi\ h$ and $c_2 \ \xi\ c_2$. The former will stabilize at the value $\frac{1}{2}$ in all models and the latter eventually stabilizes at the value $\frac{2}{3}$. Neither of these two formulas can take the value 1, so they are weakly but not strongly equivalent.

Field suggests that the strong equivalence between $\top \rightarrow \bot$ and $\bot$ is not sufficiently *uniform* to identify the properties $\lambda x.(\top \rightarrow \bot)$ and $\lambda x.\bot$. So, to attempt to interpret this suggestion, let us introduce two further senses of equivalence. Let $[\![A]\!]_{M+h}$ be the sequence $\langle |A|_{M+h_\alpha} : \alpha \in On\rangle$, where the h_α's form a revision sequence and $h_0 = h$. Suppose that we have a class of models $\mathfrak{M}$ and a class of sets of hypotheses $\mathfrak{H}$ such that for each $M \in \mathfrak{M}$ there is an associated non-empty set of initial hypotheses $\mathfrak{H}_M$. For given classes $\mathfrak{M}$ and $\mathfrak{H}$, two formulas

[23] PPC 30

[24] PPC 35

[25] If we were considering the modal operators, there would be at least one more sense of equivalence, the validity of a necessary biconditional.

$A(x)$ and $B(x)$, or two properties $\lambda x.A$ and $\lambda x.B$, are *weakly uniformly equivalent* in $\mathfrak{M}$ and $\mathfrak{H}$ iff for every $M \in \mathfrak{M}$, $h \in \mathfrak{H}_M$, and object o in the domain of M, $[\![A(o)]\!]_{M+h} = [\![B(o)]\!]_{M+h}$. Two formulas A and B are *strongly uniformly equivalent* in $\mathfrak{M}$ and $\mathfrak{H}$ iff $[\![A(o) \leftrightarrow B(o)]\!]_{M+h}$ is $\mathbf{1}$, the constant sequence of 1s, for each $M \in \mathfrak{M}$, $h \in \mathfrak{H}_M$, and object o in the domain of M. Unlike weak and strong equivalence, which ignore an initial portion of revision sequences, the uniform notions defined here consider the entirety of the revision sequences.

A pair, $\mathfrak{M}$ and $\mathfrak{H}$, provides a counterexample to the weak uniform equivalence of two properties, $\lambda x.A(x)$ and $\lambda x.B(x)$, if there is a model $M \in \mathfrak{M}$ and hypothesis $h \in \mathfrak{H}_M$ such that $[\![A(o)]\!]_{M+h} \neq [\![B(o)]\!]_{M+h}$, for some object o in the domain of M. Similarly, such a pair provides a counterexample to the strong uniform equivalence of $\lambda x.A$ and $\lambda x.B$ if for some model $M \in \mathfrak{M}$, hypothesis $h \in \mathfrak{H}_M$, and object o, $[\![A(o) \leftrightarrow B(o)]\!]_{M+h} \neq \mathbf{1}$. Any class of hypotheses that contains the hypothesis h assigning $\frac{1}{2}$ to all conditionals will provide a counterexample to the weakly uniform equivalence of $\top \rightarrow \bot$ and $\bot$, and examples to strongly uniform equivalence are easy to obtain. It seems plausible that it is a necessary condition for the identification of two properties that there not be a counterexample to their identity over a relevant class of models and initial hypotheses. However, we will see that this condition is not sufficient: a lack of counterexamples does not suffice for property identity.

In PPC footnote 16, Field suggests generating the set of reflection hypotheses in a general way and then using those to select distinguished initial hypotheses h^* for revision sequences. In our notation, $\mathfrak{H}_M$ is the set of all such distinguished initial hypotheses h^* for a model M. On this proposal, for any model M and for any hypothesis h in a revision sequence starting from any $h^* \in \mathfrak{H}_M$, $|\top \rightarrow \bot|_{M+h} = 0$, and $|\bot|_{M+h}$ is always 0. The distinguished hypotheses in $\mathfrak{H}$ then do not provide counterexamples to the identity of certain properties that must be kept distinct. Neither uniform equivalence notion is sufficient for identifying properties.

At the very least, identity between properties needs to obey certain logical principles, such as a principle of substitution of identicals. To see how Field secures this, it will be useful to recapitulate some of his discussion here. To secure the desired logical behavior one can use a binary relation R comprising three parts: an equivalence relation over properties, an equivalence relation over propositions, and identity over the other objects. Say that a hypothesis h is R-good iff for all closed property abstracts b and c such that Rbc, for all objects o, $|o \ \xi \ b|_{M+h} = |o \ \xi \ c|_{M+h}$. It is further required that all hypotheses in the revision construction be R-good.

There are models and hypotheses where such an R is a congruence relation over properties, permitting one to form a quotient model that identifies all properties R-equivalent in the original model. Say that a hypothesis is R-*congruent* iff for all closed property abstracts b and c such that Rbc, for any formulas $P(x)$ and $Q(x)$, $|P(b) \rightarrow Q(b)|_{M+h} = |P(c) \rightarrow Q(c)|_{M+h}$. Finally, a hypothesis h is strongly R-congruent iff for any formula $A(x)$, $|A(b)|_{M+h} = |A(c)|_{M+h}$. Field shows that R-good and R-congruent hypotheses are strongly R-congruent. Given that h is strongly R-congruent, $\mathcal{R}^*(h)$ will be R-congruent, and so strongly R-congruent.

While strong uniform equivalence is insufficient for property identity, Field's proof shows us a feature of revision sequences that is sufficient. Let b and c be closed property abstracts, and let R be an appropriate binary relation on a model M and an R-congruent, R-good hypothesis h. If Rbc, then $[\![A(b)]\!]_{M+h} = [\![A(c)]\!]_{M+h}$, for all formula contexts $A(x)$ with only 'x' free. This is strong uniform equivalence extended to all formulas differing on at most occurrences of b and c, which is to say indiscernibility of b and c as far as the revision sequence is concerned.

Indiscernibility across the revision sequence is enough for identifying properties. *Eventual* indiscernibility would also be sufficient.[26] Why not simply aim for eventual indiscernibility then? There is, unfortunately, no apparent reason to think that, for an appropriate R, either R-congruence or strong R-congruence will be inevitable outcomes of the revision process.

It appears that it is not the extensional features of $\top \rightarrow \bot$ and $\bot$ that create trouble for identifying the properties $\lambda x.(\top \rightarrow \bot)$ and $\lambda x.\bot$. There are classes of models and hypotheses in which those formulas are strongly uniformly equivalent and those properties are instantiated to the same degree by the same objects. The problem, rather, comes from the "intensional" features of $\top \rightarrow \bot$ and $\bot$, namely that the properties $\lambda x.(\top \rightarrow \bot)$ and $\lambda x.\bot$ may not instantiate all the same properties to the same degree.

Taking two formulas to express the same property requires them to be R-equivalent and indiscernible in some revision sequence. Field notes that formulas that are provably equivalent in S_3, symmetric Kleene logic, with the addition of

[26] Suppose $\mathcal{S}$ is a revision sequence in which b and c are eventually indiscernible. Let h be a reflection ordinal occurring at some stage after which b and c are indiscernible. The hypotheses following h will repeat, and eventually h will recur. A new sequence taking h as the initial hypothesis will be one in which only the hypotheses following h in $\mathcal{S}$ occur. In this new sequence, b and c will be indiscernible at all stages.

rules for ξ, can be taken to express the same property. What is it about provability in S_3 that ensures that? Formulas provably equivalent in S_3 can be shown to be so using rules for the logical vocabulary apart from the conditional, i.e. $\{\wedge, \vee, \neg, \forall, \lambda, \xi\}$. The conditional is the only primitive connective that is evaluated across stages of the revision construction.[27] The other logical vocabulary is evaluated at a single stage, so formulas provably equivalent in S_3 will be weakly uniformly equivalent in all revision sequences. Any R-good, R-congruent initial hypothesis can be used to obtain a revision sequence in which such formulas express indiscernible properties.

Field says that it is somewhat arbitrary how coarsely to identify properties and propositions. While this seems right for arbitrary properties and arbitrary purposes, there are some specific kinds of properties that suggest natural levels of coarse-graining. We will consider one here.

Since Field's consequence relations obey the rule of Weakening for $\rightarrow$, they are *monothetic* in the sense that all valid formulas are strongly equivalent, or to put it another way, up to strong equivalence, there is only one validity.[28] A logic with more than one validity, up to logical equivalence, is *polythetic*. A logic being monothetic suggests that it should express a unique logical property, the property of being such that logic holds. Under some plausible assumptions, one cannot have such a property in Field's naive property theory. The assumptions are that if $\lambda x.A = \lambda x.B$, then $\lambda x.\neg A = \lambda x.\neg B$ and that $\lambda x.A = \lambda x.\neg\neg A$.[29] Suppose that $\lambda x.\neg(\top \rightarrow \bot) = \lambda x.\neg\bot$. From the first assumption it follows that $\lambda x.\neg\neg(\top \rightarrow \bot) = \lambda x.\neg\neg\bot$, whence $\lambda x.(\top \rightarrow \bot) = \lambda x.\bot$, by the second assumption. As we cannot have the last identity, we cannot have the initially supposed identity.

No matter how the properties are coarse-grained, the resulting property logic, the set of coarse-grained properties of being such that some validity holds, is polythetic, in the extended sense that it has more than one member. There is, then, a mismatch between the formula logic, the valid formulas up to strong equivalence, and the property logic, the set of coarse-grained properties of being such that a certain validity holds, in the sense that the formula logic recognizes only one validity and the property logic recognizes at least two. These tell two different stories about logical truth, which we think leads to a philosophical tension.

[27]NB: We are setting aside $\rhd$.

[28]The term 'monothetic' is from Humberstone (2011, 220–221). The argument that the consequence relations are monothetic is immediate. Suppose $\models A$ and $\models B$. By Weakening, $\models B \rightarrow A$ and $\models A \rightarrow B$.

[29]Both assumptions reflect weak equivalences in S_3.

The stated purpose of talking about properties and propositions is "to provide a natural framework for talking about language and the mind."[30] One particular application for Field's theory that is of interest to us is as a framework for talking and thinking about Field's logic, and one of the features of interest is logical truth. Using Field's theory for this appears to require that we draw a distinction between some logical truths, represented as properties, that does not fit neatly with certain facts about the logic itself, namely that there is essentially only one logical truth. This tension gives us pause.

One potential way to avoid this tension, although it is one Field rejects, is to adopt a polythetic logic for the formula logic, and there are many polythetic logics from which to choose. Many relevant logics will do, including, for example, Brady's depth relevant systems, which have already been shown to allow for strong property abstraction principles. Unlike with Field's logic, in any of these relevant logics, there are many validities, up to strong equivalence. For example, even in the stronger relevant logics, $p \to p$ is not equivalent to $q \to q$. In many, but not all, $\top \to \bot$ is not strongly equivalent to $\bot$. For a range of relevant logics, then, there is no motivation for identifying the property $\lambda x.(\top \to \bot)$ with $\lambda x.\bot$. As the formulas are not strongly equivalent, they are distinct from the point of view of logic.[31]

4 Extensionality and triviality

One last point can be made concerning the range of equivalence notions available in any of the logics we have discussed here, and considering it will bring us full circle, back to the initial considerations of proof-theoretic semantics and rules of inference, where we began. Recall the point we made on page 13 concerning weak and strong equivalence. It is one thing, in the model theory, to think of A and B as equivalent if they are satisfied in the same models. It is another to think of them as equivalent if the biconditional $A \leftrightarrow B$ is logically valid. The same distinction can be made, of course, for the unidirectional notion of consequence. We can think of A as weakly entailing B if all models satisfying A also satisfy B. Or we can think of A as strongly entailing B if $A \to B$ is logically valid. These are

[30] PPC 1

[31] This is not to say there are not further issues one may have with property identity in relevant logics, some of which are raised by Field, but they do not run into the same issue that we highlighted above.

different notions, because the logic in question does not satisfy the traditional deduction theorem to the effect that $A \models B$ if and only if $\models A \rightarrow B$.

To bring our considerations full circle, we should note that the application of the distinction between weak and strong entailment to matters of *proof* and *inference* is not straightforward or immediate. If we think of inference rules like $\rightarrow E$ or $\rightarrow I$, in which consequences are drawn out or assumptions are discharged, we may not have specified these rules in terms of their behavior of *models*.[32] Should we think of these steps as corresponding to weak validity, or strong validity? This point becomes pressing when it comes to Field's response to Restall's (2010) triviality argument. That argument is presented in a sequent calculus with sequents of the form $X \Rightarrow A$ (or $X \Rightarrow$ with an empty right hand side), whose intended interpretation is consonant with the rules of proof discussed earlier in this paper: $X \Rightarrow A$ is a derivable sequent when there is a proof of A from the assumptions X (and $X \Rightarrow$ is derivable when there is a proof of $\perp$ from the assumptions X, i.e., a reduction of X to absurdity).[33]

When Field discusses Restall's argument, he takes it that a derivation of a sequent $A \Rightarrow B$ is to be understood as corresponding to a claim of weak entailment, of the form $A \models B$. That is a *permissible* reading of the argument, but it is in no way *obligatory*. If I have proved B under the assumption that A (appealing to no other assumptions), what can I learn about the relationship between A and B? It is plausible that I do learn that $A \models B$ (that in each model in which A is satisfied, so is B), if satisfaction in models is indeed closed under all of the rules of proof. If models allow for formulas to be evaluated in degrees (as Field's do), then perhaps proof tells us *more* than that satisfaction is preserved. Exactly what, we cannot tell, because Field has not given us his own account of proof or inference to go along with his models.

What more could proof tell us? There are at least four senses of entailment available, weak, strong, and their uniform counterparts. To better understand the logic and potential property identities, we need to move beyond weak entailment. We need to know when a proof from A to B suffices to establish a strong entailment, for example. And here, we suspect that the proof-theoretic approaches discussed in §1, and related work on substructural logics, may have something

[32] We *especially* need not have specified them with regard to complex models such as the revision sequences Field uses.

[33] Actually, the rules discussed in that review have a more general form $X \Rightarrow Y$ allowing for multiple conclusions, but this generalisation is not important for the triviality proof, which uses at most one conclusion in each sequent.

to offer.[34] For example, there cannot be a strong entailment from $A \wedge (A \rightarrow B)$ to B, nor can there be one from $A \rightarrow (A \rightarrow B)$ to $A \rightarrow B$ on pain of triviality. Strong entailment is a non-contractive logic, which suggests that proofs will be sensitive to how many copies of each assumption are available. Given the earlier remarks on the rule of Permutation, proofs, in particular $\rightarrow I$, will also be sensitive the order of assumptions, in a sense of 'order' appropriate to the proof theory.

Restall's (2010) triviality argument is a challenge for all notions of entailment for a theory of properties, not just for weak entailment. This is a challenge that Field's view can meet, in the sense that it is provably consistent (the trivialising conclusion is not derivable). The challenge that remains is showing how to live without the inference principles used in the paradoxical derivation, and in showing what inference principles we can find safe. The principles in question for Field's theory are the identity or non-identity conditions for properties, without which it is not clear that we have a firm grasp on what the properties are. Field's suggestion for using weak equivalence in S_3 goes some way to addressing the problem, but there is much yet to say about properties that involve the conditional in a substantive way, which are the properties that are really distinctive of the approach in PPC.

5 Conclusion

To conclude, let us take stock. In PPC, Field offers a compelling picture of naive property theory. His work has gone a long way towards showing us how to satisfy the stringent demands of a naive theory. Our response has tried to shed some light on the methodology of Field's approach and some of its philosophical features. Our response came in two parts, one concerning the development of Field's theory and the other concerning property identity. In the first part, we present an alternative, proof-theoretic approach to naive property theory, which shares some of the formal virtues of Field's approach. This approach is presented as a foil to Field's and to throw into relief some of the choices made by Field, in particular the reason for rejecting Permutation. The second part turns to the models to discuss property identity, equivalence, and the response to Restall's (2010) triviality argument. Here our purpose was to unpack Field's comment about uniform validity and its connection to property identity. In doing so, we distinguished different kinds of equivalence, which were then used in commenting on Field's

[34] For more on substructural logics, see Restall (2000).

response to Restall's triviality argument.

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