Two Traditions in Abstract Valuational Model Theory

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Abstract

We investigate two different broad traditions in the abstract valuational model theory for nontransitive and nonreflexive logics. The first of these traditions makes heavy use of the natural Galois connection between sets of (many-valued) valuations and sets of arguments. The other, originating with work by Grzegorz Malinowski on nonreflexive logics, and best systematized in Blasio et al. [2017], lets sets of arguments determine a more restricted set of valuations. After giving a systematic discussion of these two different traditions in the valuational model theory for substructural logics, we turn to looking at the ways in which we might try to compare two sets of valuations determining the same set of arguments.

1 Introduction

What is the space of possibilities for giving valuational model theory for substructural logics? In previous work we've explored one way of generalizing a common approach to the valuational model theory for fully structural logics, championed in [Scott, 1974; Shoesmith and Smiley, 1978] ([Humberstone, 2012, pp.57–59] contains a detailed introductory presentation of this perspective), which emphasizes the role of a Galois connection between consequence relations and sets of bivaluations. In this previous work, we've shown how to extend these Galois connections to logics that need not obey reflexivity or transitivity, by moving from two values to three and four. Call this the *Galois tradition*. This is not the only systematic way of giving a valuational model theory for substructural logics, though. There is another tradition, coming from works beginning with [Malinowski, 1990], and drawn on in [Blasio et al., 2017; Frankowski, 2004], which has played a central role in the development of the abstract model theory of nonreflexive and nontransitive logics. Call this other way of associating sets of valuations with sets of arguments the *Malinowski tradition*. In this paper we present a systematic and unifying treatment of this tradition, the resulting uniform treatment being broadly similar to the valuational treatment of such logics in [Blasio et al., 2017]. Our main goal here is to attempt to situate these two traditions relative to one another. To this end we investigate a variety of different ways of comparing sets of valuations.

The road-map for the present paper is as follows. We begin in section 2 by introducing the common aspects of the abstract valuational approach to logical consequence against whose background our investigation of the two traditions are defined. We then, in section 3 and section 4, introduce the two different traditions, before in section 5 looking at the various ways in which we might compare sets of valuations. An appendix then investigates the question of when a given set of arguments in the SET-SET framework has a least class of valuations determining it, complementing similar results concerning arguments in the SET-FMLA framework given in section 5.

2 Abstract Valuational Approaches to Logical Consequence

Abstract valuational approaches to logical consequence of the kind which we will be concerned with here involve the interplay between three components: some set $\mathfrak{V}$ of *valuations*, some set $\mathfrak{A}$ of *arguments*, and a binary *counterexample* relation $⋇$ from $\mathfrak{V}$ to $\mathfrak{A}$. Throughout we will regard $\mathfrak{V}$ and $\mathfrak{A}$ as determined by some *language* $\mathcal{L}$, which we will take to simply be a set, the members of which we refer to as *formulas*.[1] Note in particular that we are ignoring any structure the formulas themselves might exhibit, treating each formula alike simply as a member of $\mathcal{L}$.

The way in which our language, $\mathcal{L}$, determines our set of arguments, $\mathfrak{A}$, depends on which *logical framework*, or simply *framework*, we are working in. (We take the term from [Humberstone, 2012, pp.103–112].) There are two frameworks which are of primary interest in the present paper, each of which provides a different account of what an argument is, and thus of what the set $\mathfrak{A}$ looks like.

- According to the framework SET-FMLA an argument consists of a pair $\langle \Gamma, \phi \rangle$ of a set of formulas and a single formula, which we will write as $[\Gamma \rhd \phi]$. In the framework SET-FMLA the set of all arguments is $\mathfrak{A}_{\text{SF}} = \wp(\mathcal{L}) \times \mathcal{L}$.

- According to the framework SET-SET an argument consists of a pair $\langle \Gamma, \Delta \rangle$ of sets of formulas, which we will write as $[\Gamma \succ \Delta]$. In the framework SET-SET the set of all arguments is $\mathfrak{A}_{\text{SS}} = \wp(\mathcal{L}) \times \wp(\mathcal{L})$.

In the present paper we stick almost entirely to the framework SET-FMLA, but we will have occasional reason to consider how matters fare in SET-SET as well. When we do not explicitly remark on framework below, we are speaking in SET-FMLA. It is sometimes useful to think of arguments as ordered by the partial order $\sqsubseteq$:

Definition 1. For SET-FMLA arguments, $[\Gamma \rhd \phi] \sqsubseteq [\Gamma' \rhd \psi]$ iff $\Gamma \subseteq \Gamma'$ and $\phi = \psi$; and for SET-SET arguments, $[\Gamma \succ \Delta] \sqsubseteq [\Gamma' \succ \Delta']$ iff $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.

Consequence relations can be thought of as sets of arguments. Often, however, the phrase 'consequence relation' is understood to impose certain conditions on such a set; not every set is meant to count. (For example in [Humberstone, 2012, p. 55].) To avoid

[1] Throughout we will use ϕ, ψ and other lowercase Greek letters as schematic letters for formulas, and Γ, Δ, Σ and other uppercase Greek letters for sets of formulas.

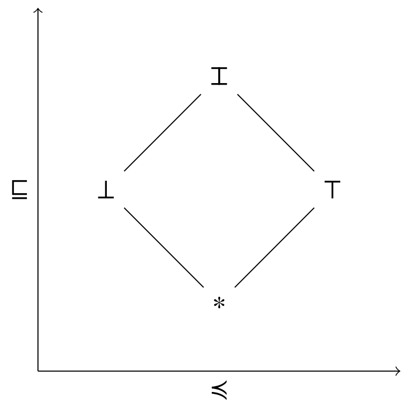

Figure 1: Two orders on values

even suggesting such assumptions, we frame our discussion entirely in terms of sets of arguments, and we make any needed restrictions explicit as we go. We expect, though, that natural applications of our results will be to sets of arguments understood as the set of *valid* arguments of some logical system or other, and this connection inspires some of our terminology.

In particular, we consider the following three conditions on sets of arguments. (Here we state them in their SET-FMLA forms, since that is our present focus. See [French and Ripley, 201X] for the appropriate SET-SET versions.)

Definition 2. A set A of SET-FMLA arguments is:

- *reflexive* iff for each $\phi \in \mathcal{L}$, $[\phi \rhd \phi] \in A$;
- *monotonic* iff whenever $a \in A$ and $a \sqsubseteq b$, then $b \in A$;
- *completely transitive* iff for all $\Sigma \subseteq \mathcal{L}$, if $[\Gamma \rhd \sigma] \in A$ for each $\sigma \in \Sigma$ and $[\Sigma, \Gamma \rhd \phi] \in A$, then $[\Gamma \rhd \phi] \in A$.

This paper focuses on connections between sets of arguments on the one hand and sets of *valuations* on the other. Following [French and Ripley, 201X] (and implicitly [Humberstone, 1988]), we work with *tetravaluations*: members of $\mathfrak{V}$, the set of functions from the language into the set $\{\top, \bot, ⌶, *\}$ of values. We consider $\{\top, \bot, ⌶, *\}$ as a *bilattice*, equipped with two distinct lattice orders $\sqsubseteq$ and $\preccurlyeq$, as depicted in fig. 1, and lift each order to $\mathfrak{V}$ itself pointwise. We refer to the order $\sqsubseteq$ as the *information order*, and to the order $\preccurlyeq$ as the *truth order*.

In this setting, we can present in a particularly abstract way the usual process of determining a set of arguments by specifying a set of models, a way that allows us to abstract away from many of the details often associated with models. This usual process depends on having some sense of what it takes for a model to be a counterexample to an argument, and then counts an argument as valid iff it has no countermodels. Here, we define our counterexample relation as in definition 3.

Definition 3. A valuation v is a *counterexample* to an argument $a = [\Gamma \rhd \phi]$ (in the SET-SET framework $a = [\Gamma \succ \Delta]$)—written $v \divideontimes a$—iff $v[\Gamma] \subseteq \{\top, \top\!\!\!\!\bot\}$ and $v(\phi) \in \{\bot, \top\!\!\!\!\bot\}$ (in the SET-SET framework $v[\Delta] \subseteq \{\bot, \top\!\!\!\!\bot\}$).

In effect, our four values simply encode every possible combination of counterexampley behaviour: $\top$ is a premise-counterexample value; $\bot$ is a conclusion-counterexample value; $\top\!\!\!\!\bot$ is both of these; and $*$ is neither. With this understanding of counterexampling in mind, any set $V \subseteq \mathfrak{V}$ uniquely determines a set of arguments $\mathcal{A}(V) \subseteq \mathfrak{A}$, namely the set of all arguments which have no counterexamples in V.

Already, this gives us some texture to work with. (For proofs of the following claims, see [French and Ripley, 201X].) First, $\mathcal{A}(V)$ is always monotonic, for any set V of valuations. Second, $\mathcal{A}(V)$ is reflexive if and only if there is no $v \in V$ and $\phi \in \mathcal{L}$ with $v(\phi) = \top\!\!\!\!\bot$. And third, $\mathcal{A}(V)$ is completely transitive if there is no $v \in V$ and $\phi \in \mathcal{L}$ with $v(\phi) = *$.[2]

3 The Galois Tradition

The connection just described between sets of valuations and monotonic sets of arguments is a strong one—keeping fixed our notion of counterexample, a given set of valuations determines a single monotonic set of arguments. But which monotonic sets of arguments can be determined in this way? As it happens, all of them.

To see this, it is useful to define a map going in the other direction; just as we have $\mathcal{A}$ to take us from sets of valuations to sets of arguments, we want a map to take us from sets of arguments to sets of valuations. This cannot be an inverse of $\mathcal{A}$, since $\mathcal{A}$ is not injective; indeed, *every* set of arguments is determined by multiple distinct sets of valuations.[3] So we are left with some choices. It is here that the two traditions we are considering differ from each other.

In the case of the Galois tradition we associate with a given set of arguments $A \subseteq \mathfrak{A}$ the set of valuations $\mathcal{V}(A) = \{v \in \mathfrak{V} | \forall a \in A, \neg(v \divideontimes a)\}$—the set of all valuations which are not counterexamples to any argument in A. (This definition would make just as much sense in a SET-SET framework, and indeed the Galois tradition often involves study of both frameworks simultaneously, including the relations between them. See for example [Shoesmith and Smiley, 1978].)

Together with the above definition of $\mathcal{A}$ (the set of arguments consistent with a given set of valuations) this instantiates a general and familiar structure: that of a Ga-

[2] Note that this third claim, unlike the second, is *not* a biconditional. As occasion arises we will refer to the classes of valuations implicitly delimited here as $\mathfrak{V}_3^r$ (for those which never assign the value $\top\!\!\!\!\bot$), $\mathfrak{V}_3^t$ (for those which never assign the value $*$). In the interests of completeness, the class of all valuations which never assign a formula either of $*$ or $\top\!\!\!\!\bot$ will be referred to as $\mathfrak{V}_2$.

[3] Since the valuation $v_\top$ that assigns $\top$ to every formula is a counterexample to *no* SET-FMLA argument, it can always be added to or removed from a set of valuations without affecting the resulting consequence relation. The same goes for any valuation that assigns only values from $\{\top, *\}$.

One particularly well-known case of this involves the SET-FMLA consequence relation of classical propositional logic. This set of arguments is determined by the usual set of all two-valued Boolean valuations (using the value $\top$ for truth and $\bot$ for falsity); but the same consequence relation is also determined by the set of valuations that adds $v_\top$—which is not Boolean—to this usual set. It is cases like this which are used in [Carnap, 1943] to motivate a shift to what is essentially the SET-SET framework. For further discussion and references on this and related issues, see [Humberstone, 2012, p.101ff].

lois connection. For any $A \subseteq \mathfrak{A}$, $V \subseteq \mathfrak{B}$, we have $A \subseteq \mathcal{A}(V)$ iff $V \subseteq \mathcal{V}(A)$; this is what it means for $\mathcal{A}$, $\mathcal{V}$ to form a Galois connection.

Galois connections are the simplification to the case of posets of the categorical notion of *adjunction*.[4] Adjunctions in general, and Galois connections in particular, crop up all over the place in mathematics, and a great deal is known about their behaviour. One of the key features of the Galois tradition is that it allows us to appeal directly to this body of work.

Theorem 1 (Galois facts)**.** *For any $V, V' \subseteq \mathfrak{B}$ and $A, A' \subseteq \mathfrak{A}$,*

(i) if $V \subseteq V'$, then $\mathcal{A}(V) \supseteq \mathcal{A}(V')$,

(ii) if $A \subseteq A'$, then $\mathcal{V}(A) \supseteq \mathcal{V}(A')$,

(iii) $\mathcal{V} \circ \mathcal{A}$ (henceforth, $\mathcal{V}\mathcal{A}$) is a closure operation on $\langle \wp(\mathfrak{B}), \subseteq \rangle$,[5]

(iv) $\mathcal{A} \circ \mathcal{V}$ (henceforth, $\mathcal{A}\mathcal{V}$) is a closure operation on $\langle \wp(\mathfrak{A}), \subseteq \rangle$,

(v) $\mathcal{V}(A)$ is closed wrt $\mathcal{V}\mathcal{A}$,

(vi) $\mathcal{A}(V)$ is closed wrt $\mathcal{A}\mathcal{V}$, and

(vii) $\mathcal{A}$ and $\mathcal{V}$ form an (order-inverting) isomorphism between the closed elements of $\wp(\mathfrak{B})$ and the closed elements of $\wp(\mathfrak{A})$.

Proof. See [Ore, 1944, p. 494–496]. Further useful discussion can be found in [Bimbó and Dunn, 2008; Birkhoff, 1967; Davey and Priestley, 2002; Dunn, 1991; Erné et al., 1993].[6] □

Another nice feature of the Galois tradition comes from its providing an *abstract soundness and completeness theorem* for all monotonic sets of arguments (a connection emphasised in, for example, [Dunn and Hardegree, 2001; Hardegree, 2005]). Say that a set A of arguments is *sound* for a set of valuations V iff $A \subseteq \mathcal{A}(V)$ (i.e. if those arguments are among the arguments consistent with that set of valuations), and *complete* for it iff $\mathcal{A}(V) \subseteq A$ (i.e. if the set of arguments contains all the arguments which are consistent with that set of valuations). As should be clear, these are by no means nonstandard uses of 'sound' and 'complete', even if they are a bit more abstract than usual. So A is sound and complete for V iff $A = \mathcal{A}(V)$—that is, when it is precisely the set of arguments with no counterexamples in that set of valuations.

It is a fact for any monotonic set A that $A = \mathcal{A}\mathcal{V}(A)$—in the parlance of the Galois tradition, every monotonic set of arguments A is *closed*. (This does *not* follow from

[4] Galois connections come in *antitone* and *monotone* versions; we are here using the (original) antitone version. These are essentially the same thing, however: a monotone Galois connection between S and T is exactly an antitone Galois connection between S and the order-dual of T. For helpful discussion on this difference, see [Dunn, 1991].

[5] A *closure* operation on a partially-ordered set $\langle S, \leq \rangle$ is an operation C such that for every $X, Y \in S$: 1) $X \leq C(X)$; 2) if $X \leq Y$, then $C(X) \leq C(Y)$; and 3) $C(C(X)) \leq C(X)$. (Equivalently, such that for every $X, Y \in S$: $X \leq C(Y)$ iff $C(X) \leq C(Y)$.) An $X \in S$ is *closed* wrt C iff $X = C(X)$.

[6] But note that [Davey and Priestley, 2002; Erné et al., 1993] use the monotone understanding of Galois connection rather than the antitone one; recall footnote 4.

$\mathcal{A}, \mathcal{V}$ forming a Galois connection; all that ensures is the soundness direction. This is an additional result. For more on this result see [French and Ripley, 201X, Section 2.1].) This is the abstract soundness and completeness theorem: it gives us a systematic way, given a monotonic set A, of giving a set of valuations that A is both sound and complete for. The set is $\mathcal{V}(A)$.

Before moving to the other tradition we consider here, we pause to note that the above presentation of the Galois tradition is a bit idiosyncratic. While the Galois tradition itself is widely-explored, this exploration has mainly (for example in [Dunn and Hardegree, 2001; Hardegree, 2005; Humberstone, 2012; Shoesmith and Smiley, 1978]) stuck to the case where only the values $\top, \bot$ are used. In that setting, all of monotonicity, reflexivity, and complete transitivity are enforced. Extending the tradition to the four-valued version we consider here is recent, and is the topic of [French and Ripley, 201X]. (It is implicit in [Humberstone, 1988].) This extension is also put to use in [Ripley, 2018].

4 Malinowski Valuations

By contrast, the other tradition we consider has been more flexible from the beginning. It was developed initially in [Malinowski, 1990], to provide a valuational grip on *nonreflexive* sets of arguments. It has since been taken up in [Blasio et al., 2017; Frankowski, 2004], to work with nontransitive sets of arguments as well.

Like the Galois tradition, this other tradition, which we call the *Malinowski tradition*, centres on the map $\mathcal{A}$ we have already met from sets of valuations to sets of arguments, and provides an additional map going the other way, from sets of arguments to sets of valuations. The difference is in this additional map; the Malinowski tradition does not use $\mathcal{V}$, but rather a different map we will call $\mathcal{M}$. Since Galois connections are uniquely specifying, there can be no map from sets of valuations to sets of arguments Galois-connected to $\mathcal{A}$ other than $\mathcal{V}$ itself. Since $\mathcal{M}$ is distinct, it is *not* Galois-connected to $\mathcal{A}$.

To define $\mathcal{M}$, we begin from the notion of a *Malinowski valuation*.

Definition 4. Given any set A of SET-FMLA arguments and set Γ of formulas, the *Malinowski valuation* m_A^Γ determined by A and Γ is the valuation such that $m_A^\Gamma(\phi) =$

- $\top$ iff $\phi \in \Gamma$ and $[\Gamma \rhd \phi] \in A$
- I iff $\phi \in \Gamma$ and $[\Gamma \rhd \phi] \notin A$
- $\bot$ iff $\phi \notin \Gamma$ and $[\Gamma \rhd \phi] \notin A$
- $*$ iff $\phi \notin \Gamma$ and $[\Gamma \rhd \phi] \in A$

This is defined as it is because of the following proposition:

Proposition 1. *$m_A^\Gamma \text{※} [\Sigma \rhd \phi]$ iff: $\Sigma \subseteq \Gamma$ and $[\Gamma \rhd \phi] \notin A$.*

Proof. LTR: Suppose that $m_A^\Gamma \text{※} [\Sigma \rhd \phi]$. This is the case iff $m_A^\Gamma[\Sigma] \subseteq \{\top, \mathrm{I}\}$ and $m_A^\Gamma(\phi) \in \{\mathrm{I}, \bot\}$. From definition 4 we can see that $m_A^\Gamma(\psi) \in \{\top, \mathrm{I}\}$ iff $\psi \in \Gamma$, and so $\Sigma \subseteq \Gamma$. Similarly, as $m_A^\Gamma(\phi) \in \{\mathrm{I}, \bot\}$, it follows that $[\Gamma \rhd \phi] \notin A$.

RTL: Suppose that $\Sigma \subseteq \Gamma$ and $[\Gamma \rhd \phi] \notin A$. It follows from definition 4 that, as $\Sigma \subseteq \Gamma$ that $m^{\Gamma}_{A}(\Sigma) \subseteq \{\top, \mathrm{I}\}$, and as $[\Gamma \rhd \phi] \notin A$ that $m^{\Gamma}_{A}(\phi) \in \{\mathrm{I}, \bot\}$, from which it follows that $m^{\Gamma}_{A} ⨳ [\Sigma \rhd \phi]$, as desired. $\square$

The next proposition illustrates a connection between Malinowski valuations and the information ordering $\sqsubseteq$ on valuations from fig. 1 (which, as mentioned above, is lifted to an ordering on $\mathfrak{V}$ pointwise).

Proposition 2. *Fix some $\Gamma \subseteq \mathcal{L}$ and $A \subseteq \mathfrak{A}$ such that for some $\chi \in \mathcal{L}$ we have $[\Gamma \rhd \chi] \notin A$. Then, for any valuation v such that for all ϕ, if $[\Gamma \rhd \phi] \notin A$ then $v ⨳ [\Gamma \rhd \phi]$, we have $m^{\Gamma}_{A} \sqsubseteq v$.*

Proof. Suppose that for all ϕ, if $[\Gamma \rhd \phi] \notin A$ then it holds that $v ⨳ [\Gamma \rhd \phi]$. Take an arbitrary formula ψ, to show that $m^{\Gamma}_{A}(\psi) \sqsubseteq v(\psi)$. We proceed by cases:

- If $\psi \in \Gamma$: By the fact that $[\Gamma \rhd \chi] \notin A$ we must have $v[\Gamma] \subseteq \{\top, \mathrm{I}\}$. So the only way $m^{\Gamma}_{A}(\psi) \not\sqsubseteq v(\psi)$ is if $m^{\Gamma}_{A}(\psi) = \mathrm{I}$ and $v(\psi) = \top$. But if $m^{\Gamma}_{A}(\psi) = \mathrm{I}$, then it must be that $[\Gamma \rhd \psi] \notin A$; and if $v(\psi) = \top$, then v is not a counterexample to $[\Gamma \rhd \psi]$, contradicting our initial supposition about v.

- If $\psi \notin \Gamma$: Then we know that $m^{\Gamma}_{A}(\psi) \in \{\bot, *\}$, so the only way $m^{\Gamma}_{A}(\psi) \not\sqsubseteq v(\psi)$ is if $m^{\Gamma}_{A}(\psi) = \bot$ and $v(\psi) \in \{\top, *\}$. Since $m^{\Gamma}_{A}(\psi) = \bot$, we know that $[\Gamma \rhd \psi] \notin A$. But if $v(\psi) \in \{\top, *\}$ then it is not a counterexample to $[\Gamma \rhd \psi]$, contradicting our initial supposition about v.

So it follows that, for every formula ψ, we have $m^{\Gamma}_{A}(\psi) \sqsubseteq v(\psi)$, and thus that $m^{\Gamma}_{A} \sqsubseteq v$ as desired. $\square$

Propositions 1 and 2 tell us that m^{Γ}_{A} is a universal counterexample to every A-invalid argument with premises Γ, and in addition that if there is any A-invalid argument with those premises, then m^{Γ}_{A} is the information-least among all such universal counterexamples.[7] Since each Malinowski valuation counterexamples all A-invalid arguments with a particular set of premises, we can be sure to have enough counterexamples by considering each set of premises in turn and collecting up all their Malinowski valuations. This is just what the map $\mathcal{M}$ does:

Definition 5. For any set A of arguments, $\mathcal{M}A = \{m^{\Gamma}_{A} \mid \Gamma \subseteq \mathcal{L}\}$.

The core of this idea is contained in the second claim of [Malinowski, 1990, Lemma 3.1]; a similar notion is defined in [Frankowski, 2008, Thm. 5]. (Neither source considers the full four-valued setup we use here; each uses a different set of three of the values.) [Blasio et al., 2017] arrives at this exact approach.[8] Note that, in contrast to the Galois tradition, there is no obvious way to extend this idea to the SET-SET framework.

[7] Note that proposition 2 does not hold in cases where Γ is nonempty and explosive according to A in the sense that for all ϕ we have $[\Gamma \rhd \phi] \in A$. In this case m^{Γ}_{A} is just a characteristic function for Γ, assigning $\top$ to all formulas in Γ and $*$ to everything else. But in this case every valuation counterexamples every A-invalid argument whose premises are Γ (there are none!), so in particular the $\sqsubseteq$-least valuation v_* which assigns $*$ to every formula does so. But as in this case (since Γ is nonempty) m^{Γ}_{A} is not v_* it is also not $\sqsubseteq$-below it.

[8] What we write as m^{Γ}_{A}, [Blasio et al., 2017] writes as $\mathcal{M}^{q}_{\Gamma}$, leaving A implicit; and what we write as $\mathcal{M}A$ they write as $\mathcal{B}^{q}_{A}$. See [Blasio et al., 2017, p. 239].

(Indeed, when [Blasio et al., 2017] moves from Set-Fmla to Set-Set, it also moves from the Malinowski to the Galois tradition, without commenting on the change.)

One useful result in working with Malinowski valuations is the following, which allows us to import results from the Galois tradition in reasoning about Malinowski valuations.

Proposition 3. *A is monotonic iff $\mathcal{M}A \subseteq \mathcal{V}A$.*

Proof. LTR: We prove the contrapositive. So take any A, and suppose there is some $m^{\Gamma}_{A} \notin \mathcal{V}A$; then there is some $[\Sigma \rhd \phi] \in A$ with $m^{\Gamma}_{A} \divideontimes [\Sigma \rhd \phi]$. We must have $m^{\Gamma}_{A}(\phi) \in \{\overline{\bot}, \bot\}$, and so $[\Gamma \rhd \phi] \notin A$. But by proposition 1, $\Sigma \subseteq \Gamma$, and so A must not be monotonic.

RTL: Again, we show the contrapositive. Suppose A is not monotonic; then there are $[\Gamma \rhd \phi] \in A$ and $[\Gamma' \rhd \phi] \notin A$ with $\Gamma \subseteq \Gamma'$. Since $[\Gamma' \rhd \phi] \notin A$, we have $m^{\Gamma'}_{A} \divideontimes [\Gamma' \rhd \phi]$. But then by proposition 1, $m^{\Gamma'}_{A} \divideontimes [\Gamma \rhd \phi]$, and since $[\Gamma \rhd \phi] \in A$, this means $m^{\Gamma'}_{A} \notin \mathcal{V}A$. □

One immediate similarity to the Galois tradition is in abstract soundness and completeness. For any monotonic A, we have $A = \mathcal{AM}(A)$, just as we had $A = \mathcal{AV}(A)$. So, just as the Galois tradition does, the Malinowski tradition allows us to give a valuational presentation of any monotonic set of arguments.

Theorem 2. *If A is monotonic, then $\mathcal{AM}(A) = A$. (See [Malinowski, 1990, Thm. 3.2(i)], [Frankowski, 2008, Thm. 5], [Blasio et al., 2017, Thm. 1].)*

Proof. Suppose A is monotonic. Then by proposition 3, $\mathcal{M}(A) \subseteq \mathcal{V}(A)$, so $\mathcal{A}(\mathcal{V}(A)) \subseteq \mathcal{A}(\mathcal{M}(A))$, and thus $A \subseteq \mathcal{A}(\mathcal{M}(A))$. Conversely, if $[\Gamma \rhd \phi] \notin A$, then $m^{\Gamma}_{A} \divideontimes [\Gamma \rhd \phi]$ by proposition 1, and $m^{\Gamma}_{A} \in \mathcal{M}(A)$. So $\mathcal{A}(\mathcal{M}(A)) \subseteq A$. □

So, much like the Galois tradition, the Malinowski tradition gives us a systematic way, given a monotonic set of arguments A, of providing a set of valuations that A is both sound and complete for—in this case, the set being $\mathcal{M}(A)$.

5 Comparing Sets of Valuations

As we've seen in the previous two sections, the Galois and Malinowski traditions both provide us with a method for systematically associating a given monotonic set of arguments with a set of valuations in such a way that we can prove an abstract soundness and completeness result. What can we say about the relationships which hold between these two different classes of valuations? What we will do here is to compare these two different sets of valuations by comparing them along the following, relatively natural, dimensions.

- **Inclusion.** What relationships do the two sets of valuations stand in considered as sets?
- **Frameworks.** Given that the Set-Set-framework is more expressive than the Set-Fmla-framework, how do the sets of Set-Set arguments which the two sets of valuations determine compare?

- **Order Relations.** How are the two sets of valuations related by our two ordering relations on valuations: the information order $\sqsubseteq$, and the truth order $\preccurlyeq$?
- **Position in the Lattice of Classes of Valuations Determining** A**.** Do these two sets of valuations have interesting properties when considered as members of $Mod(A)$—the collection of all sets of valuations determining A?

This is, of course, not an exhaustive list of the different respects in which we could compare sets of valuations, but it will be enough for us to at least be able to glimpse the fine texture of how the two traditions are related.

5.1 Inclusion

We will begin with the most natural way of comparing these two different classes of valuations, namely as sets. Here we have already seen in proposition 3 that $\mathcal{M}A \subseteq \mathcal{V}A$ iff A is monotonic. The reason for this link to monotonicity is interesting, and allows us to highlight an relationship between the Galois and Malinowski traditions. For any set A of arguments, we have $\mathcal{AM}(A) \subseteq A \subseteq \mathcal{AV}(A)$; and since $\mathcal{A}$ can only ever deliver monotonic sets of arguments, when A itself is not monotonic these subset relations are proper. Indeed, just as $\mathcal{AV}(A)$ is A's monotonic *closure*, $\mathcal{AM}(A)$ is its monotonic *interior*. (See theorem 3.) This is a less familiar notion than monotonic closure, but just as well-defined: the monotonic interior of A is the monotonic $B \subseteq A$ such that all monotonic $C \subseteq A$ are subsets of B; just as the monotonic closure of A is the monotonic $B \supseteq A$ such that all monotonic $C \supseteq A$ are supersets of B.

Theorem 3. *$\mathcal{AM}(A) = \{[\Sigma \rhd \phi] \mid [\Gamma \rhd \phi] \in A$, for every $\Gamma \supseteq \Sigma\}$*

Proof. $[\Sigma \rhd \psi] \in \mathcal{AM}(A)$ iff there is no $\Gamma \subseteq \mathcal{L}$ with $m_A^\Gamma \divideontimes [\Sigma \rhd \psi]$. By proposition 1, this holds iff there is no $\Gamma \supseteq \Sigma$ with $m_A^\Gamma \divideontimes [\Sigma \rhd \psi]$. Since each such $m_A^\Gamma[\Sigma] \subseteq \{\top, \mathrm{I}\}$, this in turn holds iff there is no $\Gamma \supseteq \Sigma$ with $m_A^\Gamma(\psi) \in \{\bot, \mathrm{I}\}$. And this holds iff for every $\Gamma \supseteq \Sigma$, $[\Gamma \rhd \psi] \in A$, which is what we need. □

5.2 Looking to SET-SET Counterparts

We take the idea of counterparts from [Shoesmith and Smiley, 1978, p. 72]:[9]

Definition 6. A set A of SET-FMLA arguments and a set B of SET-SET arguments are *counterparts* iff for every Γ, ϕ we have $[\Gamma \rhd \phi] \in A$ iff $[\Gamma \succ \phi] \in B$.

Given a set V of valuations, let $\mathcal{A}_{\mathrm{SS}}(V)$ be the set of SET-SET arguments with no counterexample in V. Then it is quick to see that $\mathcal{A}(V)$ and $\mathcal{A}_{\mathrm{SS}}(V)$ are always counterparts. To explore sets of valuations, $\mathcal{A}_{\mathrm{SS}}$ is more discerning than $\mathcal{A}$: there are sets V, V' such that $\mathcal{A}(V) = \mathcal{A}(V')$ but $\mathcal{A}_{\mathrm{SS}}(V) \neq \mathcal{A}_{\mathrm{SS}}(V')$. The reverse is never the case.

Every monotonic A has at least one monotonic SET-SET counterpart: recall that any such A is $\mathcal{AV}(A)$, and consider $\mathcal{A}_{\mathrm{SS}}\mathcal{V}(A)$. But some have only one, and some have

[9] This is not the definition given there, but amounts to the same and is more convenient for our purposes here.

more. As it turns out we can precisely isolate the conditions under which a set of monotonic SET-FMLA arguments has a unique SET-SET counterpart, and thus the conditions under which a set of arguments is compatible with a range of SET-SET counterparts. Say that a set Γ of formulas is *A-explosive* iff we have $[\Gamma \rhd \phi] \in A$ for every formula ϕ. Then we have the following result:

Proposition 4. *Suppose that a set A of monotonic SET-FMLA arguments is such that*

- *no set of formulas is A-explosive, and*
- *there is at most one formula ϕ with $[\rhd\phi] \notin A$*

Then A has at most one monotonic SET-SET counterpart.

Proof. To establish that an A meeting our conditions has at most one such counterpart, note first that the 'at most' in the second condition can, in light of the first condition, be treated as an 'exactly'. (If there were no such ϕ, every set Γ would be A-explosive.) Consider any monotonic SET-SET counterparts $\boldsymbol{B}$ and $\boldsymbol{C}$ of A.

- Neither $\boldsymbol{B}$ nor $\boldsymbol{C}$ can include any empty conclusion arguments. Suppose one did; say $[\Gamma \succ] \in \boldsymbol{B}$. Then since $\boldsymbol{B}$ is monotonic, $[\Gamma \succ \phi] \in \boldsymbol{B}$; and since $\boldsymbol{B}$ is a counterpart of A, that would mean $[\Gamma \rhd \phi] \in A$. But every $\psi \neq \phi$ already has $[\rhd\psi] \in A$, and since A is monotonic, that means $[\Gamma \rhd \psi] \in A$ for each of these. So Γ is A-explosive, and we have a contradiction. So $\boldsymbol{B}$ and $\boldsymbol{C}$ agree on empty conclusion arguments, by excluding them all.
- $\boldsymbol{B}$ and $\boldsymbol{C}$ must agree with A, and hence with each other, on single conclusion arguments.
- $\boldsymbol{B}$ and $\boldsymbol{C}$ must include all arguments $[\Gamma \succ \Delta]$ with $|\Delta| \geq 2$. Each such argument contains some conclusion $\psi \neq \phi$, and as we have $[\rhd\psi]$ for all such formulas $\boldsymbol{B}$ and $\boldsymbol{C}$ must both, as counterparts of A, contain the argument $[\succ\psi]$ and thus by monotonicity $[\Gamma \succ \Delta]$.

So $\boldsymbol{B}$ and $\boldsymbol{C}$ must agree everywhere, and so are identical. □

Sets of arguments which meet this condition are, admittedly, rather strange—they contain a single formula ϕ such that an argument is in that set just in case its conclusion isn't ϕ. It turns out that these strange cases are the *only* sets of arguments which have a unique monotonic SET-SET counterpart; in every other case, the SET-SET framework gives us a properly finer grip on our sets of valuations.

To see this, we need the notion of an *exact counterexample*:

Definition 7. Given a SET-FMLA argument $a = [\Gamma \rhd \psi]$, its *exact counterexample* v_a is the valuation such that $v_a(\psi) = \bot$ iff $\psi \notin \Gamma$ and I iff $\psi \in \Gamma$, and for all formulas ϕ other than ψ, $v_a(\phi) = \top$ iff $\phi \in \Gamma$ and $*$ iff $\phi \notin \Gamma$.

Proposition 5. *Suppose that a set A of monotonic SET-FMLA arguments is such that either:*

- *there is some A-explosive set of formulas Γ, or*

- *there are distinct formulas ϕ and ψ with $[\rhd\phi], [\rhd\psi] \notin A$.*

Then there are sets V_1, V_2 of valuations with $\mathcal{A}(V_1) = \mathcal{A}(V_2) = A$ but $\mathcal{A}_{ss}(V_1) \neq \mathcal{A}_{ss}(V_2)$.

Proof. Let $V_1 = \{v_a | a \notin A\}$. First, we verify that $\mathcal{A}(V_1) = A$. To see that $\mathcal{A}(V_1) \subseteq A$, take any argument $a \notin A$, and note that v_a is a counterexample to a. To see that $A \subseteq \mathcal{A}(V_1)$, take any argument $[\Gamma \rhd \phi] \notin \mathcal{A}(V_1)$. There must be some argument $b \notin A$ with $v_a ※ [\Gamma \rhd \phi]$. That is, v_a must assign everything in Γ some value from $\{\top, \mathrm{I}\}$ and ϕ either $\bot$ or I. But by definition 7, this can only happen when $[\Gamma \rhd \phi] \sqsubseteq a$. Since A is monotonic and $a \notin A$, it must be that $[\Gamma \rhd \phi] \notin A$.

So much for V_1. To find V_2:

- If there is an A-explosive set Γ, then let v_Γ be the valuation which assigns $\top$ to every $\gamma \in \Gamma$, and $*$ to all other formulas, and let $V_2 = V_1 \cup \{v_\Gamma\}$. Clearly $v_\Gamma ※ [\Gamma \succ]$. It remains to be shown that (i) $[\Gamma \succ] \in \mathcal{A}_{ss}(V_1)$ and (ii) $\mathcal{A}(V_2) = A$. For (i) note that the exact counterxample to an argument $[\Sigma \rhd \sigma]$ is a counterexample to $[\Gamma \succ]$ iff $\Gamma \subseteq \Sigma$. Since Γ is A-explosive (and A is monotonic), every superset of Γ is also A-explosive, so we cannot have $\Gamma \subseteq \Sigma$ if $[\Sigma \rhd \sigma] \notin A$. Thus there is no counterexample to $[\Gamma \succ]$ in V_1. For (ii) to fail, there would need to be some $[\Sigma \rhd \sigma] \in A$ where $v_\Gamma ※ [\Sigma \rhd \sigma]$. But v_Γ is not a counterexample to any Set-Fmla argument since it assigns no formulas $\bot$ or I.

- If there are distinct ϕ and ψ with $[\rhd\phi], [\rhd\psi] \notin A$, let $v_{\phi,\psi}$ be the valuation which assigns $\bot$ to ϕ and ψ, and $*$ to all other formulas, and let $V_2 = V_1 \cup \{v_{\phi,\psi}\}$. Clearly, $v_{\phi,\psi} ※ [\succ \phi, \psi]$. It remains to be shown that (i) $[\succ \phi, \psi] \in \mathcal{A}_{ss}(V_1)$, and (ii) $\mathcal{A}(V_2) = A$. For (i) note that exact counterexamples to Set-Fmla arguments never assign values from $\{\bot, \mathrm{I}\}$ to more than one formula, so no exact counterexample to a Set-Fmla argument is a counterexample to this argument. For (ii) to fail, there would need to be some $[\Gamma \rhd \theta] \in A$ where $v_{\phi,\psi} ※ [\Gamma \rhd \theta]$. But the only Set-Fmla arguments which $v_{\phi,\psi}$ counterexamples are $[\rhd\phi]$ and $[\rhd\psi]$, and we know that neither of those arguments are in A.

□

It is instructive to very briefly compare what we have shown here with what is known about the case of sets of monotonic, reflexive and completely transitive sets of arguments. In [Shoesmith and Smiley, 1978, p.73] it is shown that *every* monotonic, reflexive and completely transitive set of Set-Fmla arguments has at least two Set-Set counterparts, while the above results show that this does *not* fully generalise to all monotonic sets of Set-Fmla arguments. As it turns out, however, reflexivity is enough to guarantee the condition of proposition 5; transitivity doesn't seem to be involved here.

Corollary 1. *Suppose that A is a monotonic and reflexive set of arguments. Then there are sets V_1, V_2 of valuations with $\mathcal{A}(V_1) = \mathcal{A}(V_2) = A$ but $\mathcal{A}_{ss}(V_1) \neq \mathcal{A}_{ss}(V_2)$.*

Proof. Every set A of monotonic and reflexive arguments must meet one of the two conditions on proposition 5.

To see this, suppose that there is at most one formula ϕ for which $[\rhd\phi] \notin A$. If there isn't any such ϕ, then the empty set is A-explosive and we're done. If there is such a ϕ, by monotonicity and the fact that $[\rhd\psi] \in A$ for every $\psi \neq \phi$, it follows that $[\phi \rhd \psi] \in A$ for all $\psi \neq \phi$, and by reflexivity $[\phi \rhd \phi] \in A$. So it follows that $\{\phi\}$ is A-explosive.

□

What the above results tell us is that looking to the connections between sets of valuations and the SET-SET arguments brings into view differences which we cannot register by just looking at SET-FMLA arguments, for almost any A we might care about. As it turns out, considering the SET-SET framework helps us to see stronger connections between $\mathcal{V}$ and $\mathcal{M}$. Not only do they both give us a way to get a valuational grip on any monotonic SET-FMLA set of arguments, they do so in a way that matches perfectly even when extended to the more discriminating SET-SET framework:

Theorem 4. *For any set A of arguments $\mathcal{A}_{ss}\mathcal{M}(A) \subseteq \mathcal{A}_{ss}\mathcal{V}(A)$. If A is monotonic, then $\mathcal{A}_{ss}\mathcal{V}(A) = \mathcal{A}_{ss}\mathcal{M}(A)$.*

Proof. First claim: Suppose $[\Gamma \succ \Delta] \notin \mathcal{A}_{ss}\mathcal{V}(A)$. Then there is some valuation $v \in \mathcal{V}A$ with $v ※ [\Gamma \succ \Delta]$. For every $\delta \in \Delta$, this gives $v ※ [\Gamma \rhd \delta]$, so $[\Gamma \rhd \delta] \notin A$. This in turn means that for every $\delta \in \Delta$, $m^\Gamma_A(\delta) \in \{\mathrm{I}, \bot\}$. And we know that $m^\Gamma_A[\Gamma] \subseteq \{\top, \mathrm{I}\}$; so $m^\Gamma_A ※ [\Gamma \succ \Delta]$. Thus, $[\Gamma \succ \Delta] \notin \mathcal{A}_{ss}\mathcal{M}(A)$.

Second claim: Assume A is monotonic. Then by proposition 3, $\mathcal{M}A \subseteq \mathcal{V}A$, and so by (the SET-SET-analogue of) theorem 1(ii) it follows that $\mathcal{A}_{ss}\mathcal{V}(A) \subseteq \mathcal{A}_{ss}\mathcal{M}(A)$. □

So $\mathcal{V}$ and $\mathcal{M}$ are not just any ways to get at a set of SET-FMLA arguments; the different sets of valuations they deliver always select the same SET-SET counterpart to the original SET-FMLA set.

5.3 Orderings on Valuations

Another way of comparing Galois and Malinowski valuations is by looking at how they interact with the information and truth orderings, both individually and collectively. We begin by looking at how Malinowski valuations interact with the information ordering, before going on to see how sets of Malinowski and Galois valuations are related by (the appropriate lifting of) the information and truth orderings.

Firstly, note that moving to stronger collections of arguments moves each Malinowski valuation down in the information order.

Lemma 2. *$A \subseteq A'$ iff for every $\Gamma \subseteq \mathcal{L}$, $m^\Gamma_{A'} \sqsubseteq m^\Gamma_A$*

Proof. LTR: By noting that $\top \sqsubseteq \mathrm{I}$ and $* \sqsubseteq \bot$.

RTL: Suppose that the right hand side holds, and that $[\Sigma \rhd \phi] \notin A'$ for some Σ, ϕ. Then $m^\Sigma_{A'} ※ [\Sigma \rhd \phi]$, so since $m^\Sigma_{A'} \sqsubseteq m^\Sigma_A$, we have $m^\Sigma_A ※ [\Sigma \rhd \phi]$. But this must mean $m^\Sigma_A(\phi) \in \{\mathrm{I}, \bot\}$, which holds only when $[\Sigma \rhd \phi] \notin A$. □

This result gives us the necessary tools to prove an analogue of theorem 1(ii), in this case in terms of the information order, rather than subset ordering on valuations.

To state this we will first need to lift the $\sqsubseteq$ so that it relates, not just pairs of individual valuations, but pairs of sets of valuations. To do this, we use what in [Brink, 1993, p.184] is called $\sqsubseteq^+$—the power relational analogue of $\sqsubseteq$. More generally, given a relation R between valuations, and sets of valuations V and V' let us say that $V\ R^+\ V'$ iff (i) for every $v \in V$ there is a $v' \in V'$ such that $v\ R\ v'$, and (ii) for every $v' \in V$ there is a $v \in V$ such that $v\ R\ v'$. Then the analogue we have of theorem 1(ii) is the following.

Theorem 5. *If A is monotonic, then $A \subseteq A'$ iff $\mathcal{M}A' \sqsubseteq^+ \mathcal{M}A$.*

Proof. LTR: Suppose that $A \subseteq A'$. Suppose that $v' \in \mathcal{M}A'$. Such a $v' = m^{\Gamma}_{A'}$ for some Γ, and by lemma 2 it follows that $v' \sqsubseteq m^{\Gamma}_{A}$ which (by definition) is in $\mathcal{M}A$. Suppose then that $v \in \mathcal{M}A$. Then $v = m^{\Gamma}_{A}$ for some Γ, and by lemma 2 it follows that $m^{\Gamma}_{A'} \sqsubseteq v$ and $m^{\Gamma}_{A'} \in \mathcal{M}A'$ by definition. So $\mathcal{M}A' \sqsubseteq^+ \mathcal{M}A$, as desired.

RTL: Suppose that $\mathcal{M}A' \sqsubseteq^+ \mathcal{M}A$, and consider any argument $[\Gamma \rhd \phi] \notin A'$. (If there is no such argument, we're done.) Then $m^{\Gamma}_{A'} ⨳ [\Gamma \rhd \phi]$. By our supposition, then, there is some $m^{\Sigma}_{A} \in \mathcal{M}A$ s.t. $m^{\Gamma}_{A'} \sqsubseteq m^{\Sigma}_{A}$. It follows that $m^{\Sigma}_{A} ⨳ [\Gamma \rhd \phi]$ and so by proposition 1 $\Gamma \subseteq \Sigma$ and $[\Sigma \rhd \phi] \notin A$. Since A is monotonic—this is the only place in the proof this assumption is used—$[\Gamma \rhd \phi] \notin A$, as desired. So $A \subseteq A'$. $\square$

Let us turn now to looking at how the information order relates Galois valuations and Malinowski valuations.

Definition 8. Given a valuation v, its *Malinowski premise set* M_v is $\{\phi \mid v(\phi) \in \{\top, \mathrm{I}\}\}$.

Lemma 3. *Given a monotonic set A of arguments and a $v \in \mathcal{V}A$, we have $v \sqsubseteq m^{M_v}_{A}$.*

Proof. Suppose $v \not\sqsubseteq m^{M_v}_{A}$. Then there is some ϕ with $v(\phi) \not\sqsubseteq m^{M_v}_{A}(\phi)$. Either $\phi \in M_v$ or not. If it is, then $v(\phi) \in \{\top, \mathrm{I}\}$, and $m^{M_v}_{A}(\phi) \in \{\top, \mathrm{I}\}$. So $v(\phi) = \mathrm{I}$ and $m^{M_v}_{A}(\phi) = \top$. By this last, $[M_v \rhd \phi] \in A$; and so $v \notin \mathcal{V}A$. If $\phi \notin M_v$, then $v(\phi) \in \{\bot, *\}$, and $m^{M_v}_{A}(\phi) \in \{\bot, *\}$. So $v(\phi) = \bot$ and $m^{M_v}_{A}(\phi) = *$. By this last, $[M_v \rhd \phi] \in A$; and so $v \notin \mathcal{V}A$. $\square$

So every valuation consistent with a given set of arguments is information-below some Malinowksi valuation for that set of arguments, namely the Malinowski valuation determined by that valuation's premise set. This gives us all the ingredients we need for the following result.

Theorem 6. *If A is monotonic, then $\mathcal{V}A \sqsubseteq^+ \mathcal{M}A$.*

Proof. Suppose that $v \in \mathcal{V}A$. By lemma 3 it follows that $v \sqsubseteq m^{M_v}_{A}$, and $m^{M_v}_{A} \in \mathcal{M}A$. Suppose, then, that $v' \in \mathcal{M}A$. As A is monotonic by proposition 3 it follows that $v' \in \mathcal{V}A$ and $v' \sqsubseteq v'$. $\square$

Interestingly lemma 3 also provides us with enough information to see how $\mathcal{V}A$ and $\mathcal{M}A$ are related by the truth ordering.

Theorem 7. *If A is monotonic, then $\mathcal{M}A \preccurlyeq^+ \mathcal{V}A$.*

Proof. Take any $m^{\Gamma}_A \in \mathcal{M}A$. Since A is monotonic, we have $\mathcal{M}A \subseteq \mathcal{V}A$ by proposition 3. So $m^{\Gamma}_A \in \mathcal{V}A$, and as $m^{\Gamma}_A \preccurlyeq m^{\Gamma}_A$, we're halfway there.

For the other half, take any $v \in \mathcal{V}A$, and consider $m^{M_v}_A$. The two valuations assign a value in $\{\top, \mathbb{I}\}$ to exactly the same formulas, and by lemma 3, $v \sqsubseteq m^{M_v}_A$. So for any formula ϕ, there are three possibilities: either $v(\phi) = m^{M_v}_A(\phi)$, or $v(\phi) = *$ and $m^{M_v}_A(\phi) = \bot$, or $v(\phi) = \top$ and $m^{M_v}_A(\phi) = \mathbb{I}$. In any of these three cases, though, $m^{M_v}_A(\phi) \preccurlyeq v(\phi)$. So $m^{M_v}_A \preccurlyeq v$, and thus there is a valuation m in $\mathcal{M}A$ with $m \preccurlyeq v$. □

The previous two theorems paint an interesting picture of how the two traditions of valuations are related. For monotonic sets of arguments, the Malinowski approach produces sets of valuations higher in the (power relation of the) information order, while the Galois approach produces sets of valuations higher in the (power relation of the) truth order.

5.4 Position in the Lattice of Classes of Valuations Determining A

Multiple different classes of valuations can determine the same set of arguments—the Galois and Malinowski valuations being just two examples. One way of thinking about how these two classes of valuations are related is to consider how they sit amongst the collection $Mod(A)$ of all A-determining classes of valuations.[10] One obvious reason for wondering about any potential connection here is the fact that the Galois valuations fill a natural place in this collection.

Proposition 6. *$\mathcal{V}A$ is the $\subseteq$-maximum element of $Mod(A)$.*

For monotonic A, we know from proposition 3 that $\mathcal{M}A$ is a member of $Mod(A)$. One natural question is whether it might be in some sense minimal in $Mod(A)$. As it turns out, in the Set-Fmla-framework at least, we are essentially never guaranteed to be able to find least sets of valuations consistent with a given set of arguments. In the remainder of this section we demonstrate that this is the case, leaving the discussion of the Set-Set cases, where least sets of valuations are much easier to come by, to the appendix. We begin by looking at what happens when we are considering bivaluations. Second, we turn to the general situation involving tetravaluations. We follow up by considering in turn both reflexive, as well as transitive trivaluations.

5.4.1 Bivaluations

We can use [Shoesmith and Smiley, 1978]'s work to help with the situation for least sets of *bivaluations* in the Set-Fmla framework. In this section, we talk of Reflexive, Monotonic, and completely Transitive sets of arguments as 'RMT' sets, to avoid taking up too much space.

Proposition 7. *If A is a Set-Fmla set of arguments and B is an RMT Set-Set set of arguments, then A and B are counterparts iff $A = \mathcal{A}_{SF}\mathcal{V}_2 B$.*

[10] That is to say $Mod(A) = \{V \mid A = \mathcal{A}(V)\}$.

Proof. LTR: suppose $A \neq \mathcal{A}_{\text{SF}}\mathcal{V}_2 B$, to show that A and B are not counterparts. Then there must be either some $[\Gamma \rhd \phi] \in A$ but not in $\mathcal{A}_{\text{SF}}\mathcal{V}_2 B$, or else some $[\Sigma \rhd \psi] \in \mathcal{A}_{\text{SF}}\mathcal{V}_2 B$ but not in A. In the first case, since $[\Gamma \rhd \phi] \notin \mathcal{A}_{\text{SF}}\mathcal{V}_2 B$, there must be some $v \in \mathcal{V}_2 B$ with $v ※ [\Gamma \rhd \phi]$. By the definition of ※, though, we have $v ※ [\Gamma \succ \phi]$, and so $[\Gamma \succ \phi] \notin B$. Thus, A and B are not counterparts. In the second case, since $[\Sigma \rhd \psi] \in \mathcal{A}_{\text{SF}}\mathcal{V}_2 B$, there must be no $v \in \mathcal{V}_2 B$ with $v ※ [\Sigma \rhd \psi]$. So, by the definition of ※, there is no $v \in \mathcal{V}_2 B$ with $v ※ [\Sigma \succ \psi]$. Thus, $[\Sigma \succ \psi] \in \mathcal{A}_{\text{SS}}\mathcal{V}_2 B$. But since B is RMT, $B = \mathcal{A}_{\text{SS}}\mathcal{V}_2 B$. So $[\Sigma \succ \psi] \in B$, and so A and B are not counterparts.

RTL: Suppose $A = \mathcal{A}_{\text{SF}}\mathcal{V}_2 B$. Now consider any $[\Gamma \rhd \phi]$. This is in A iff it has no counterexample in $\mathcal{V}_2 B$, by our supposition. This is true iff $[\Gamma \succ \phi]$ has no counterexample in $\mathcal{V}_2 B$. And this, in turn, is true iff $[\Gamma \succ \phi] \in \mathcal{A}_{\text{SS}}\mathcal{V}_2 B$. And (since B is RMT, we know $B = \mathcal{A}_{\text{SS}}\mathcal{V}_2 B$, and so) this holds iff $[\Gamma \succ \phi] \in B$. Thus, A and B are counterparts. □

Corollary 4. *If there is a set V of bivaluations with $A = \mathcal{A}_{\text{SF}}V$ and $B = \mathcal{A}_{\text{SS}}V$, then A and B are counterparts.*

Proof. $V = \mathcal{V}_2\mathcal{A}_{\text{SS}}V$, so $A = \mathcal{A}_{\text{SF}}\mathcal{V}_2\mathcal{A}_{\text{SS}}V$. Since $B = \mathcal{A}_{\text{SS}}V$, this gives $A = \mathcal{A}_{\text{SF}}\mathcal{V}_2 B$. From here, proposition 7 finishes the job. □

Theorem 8. *If A is an RMT set of* Set-Fmla *arguments, then there is a least V such that $A = \mathcal{A}_{\text{SF}}V$ iff there is a* greatest *set of arguments among the RMT* Set-Set *counterparts to A.*

Proof. LTR: Suppose there is a least such V; we claim $\mathcal{A}_{\text{SS}}V$ is greatest among RMT counterparts. It is quick to see that $\mathcal{A}_{\text{SS}}V$ is indeed an RMT counterpart: since it is determined by a set of bivaluations, it must be RMT, and since $V = \mathcal{V}_2\mathcal{A}_{\text{SS}}V$, we have $A = \mathcal{A}_{\text{SF}}\mathcal{V}_2\mathcal{A}_{\text{SS}}V$, so by proposition 7 it is a counterpart of A.

It remains to show only that for any RMT counterpart B to A we have $B \subseteq \mathcal{A}_{\text{SS}}V$. So consider any such RMT counterpart B. Since it is a counterpart, by proposition 7, $A = \mathcal{A}_{\text{SF}}\mathcal{V}_2 B$. And since V is least among sets U of valuations with $\mathcal{A}_{\text{SF}}U = A$, this means $V \subseteq \mathcal{V}_2 B$, and so (by antitonicity) $\mathcal{A}_{\text{SS}}\mathcal{V}_2 B \subseteq \mathcal{A}_{\text{SS}}V$. But since B is RMT, $B = \mathcal{A}_{\text{SS}}\mathcal{V}_2 B$, so $B \subseteq \mathcal{A}_{\text{SS}}B$.

RTL: Suppose there is a greatest RMT counterpart B to A; we claim $\mathcal{V}_2 B$ is least among sets U of valuations with $\mathcal{A}_{\text{SF}}U = A$. Since B is an RMT counterpart to A, by proposition 7 $\mathcal{A}_{\text{SF}}\mathcal{V}_2 B = A$.

It remains only to show that for any set U of bivaluations with $\mathcal{A}_{\text{SF}}U = A$ we have $\mathcal{V}_2 B \subseteq U$. So consider any U with $\mathcal{A}_{\text{SF}}U = A$. By corollary 4 $\mathcal{A}_{\text{SS}}U$ must be a counterpart to A, and since U contains only bivaluations $\mathcal{A}_{\text{SS}}U$ is RMT. Since B is greatest among RMT counterparts to A, we have $\mathcal{A}_{\text{SS}}U \subseteq B$, and so (by antitonicity) $\mathcal{V}_2 B \subseteq \mathcal{V}_2\mathcal{A}_{\text{SS}}U$. But since U is a set of bivaluations, $U = \mathcal{V}_2\mathcal{A}_{\text{SS}}U$, and so $\mathcal{V}_2 B \subseteq U$. □

Theorem 9. *If A is a compact RMT set of* Set-Fmla *arguments, then there is a least set V with $A = \mathcal{A}_{\text{SF}}V$.*

Proof. [Shoesmith and Smiley, 1978, Thm 5.11] shows that a compact RMT Set-Fmla A has a greatest RMT Set-Set counterpart. From there, apply theorem 8. □

5.4.2 Tetravaluations

Unlike the bivaluational case, when it comes to tetravaluations there is in general no guarantee that there will be a least set of valuations determining a given SET-FMLA set of arguments, even if the set of arguments obeys quite restrictive conditions. Indeed, as we will see, it is not even enough to consider sets of arguments that are both RMT and compact.

Here is an example: consider a small language $\mathcal{L} = \{p, q, r\}$, and the set of arguments A, where A is the reflexive and monotonic closure of $\{[p, q \rhd r]\}$, which is to say $A = \mathcal{A}_{\text{SF}}\mathcal{V}_3^r\{[p, q \rhd r]\}$.[11]

Theorem 10. *For this A, there is no least set V of tetravaluations such that $\mathcal{A}_{\text{SF}}V = A$.*

Proof. Let $v_{x;y;z}$ be the valuation assigning the value x to p, the value y to q, and the value z to r, and consider the following two sets. The first, V_1, is $\{v_{\top;*;\bot}, v_{*;\top;\bot}, v_{\top;\bot;\top}, v_{\bot;\top;\top}\}$. The second, V_2, is $\{v_{\top;\bot;\bot}, v_{\bot;\top;\bot}, v_{\top;\bot;\top}, v_{\bot;\top;\top}\}$. (They differ in their first two members.)

Neither contains a counterexample to $[p, q \rhd r]$, since only $v_{\top;\top;\bot}$ is such a counterexample. So $\mathcal{A}V_i \subseteq A$, for $i = 1, 2$.

To make sure each set contains enough counterexamples, it suffices to check four arguments: $[p \rhd r], [q \rhd r], [p, r \rhd q]$, and $[q, r \rhd p]$; every argument not in A is a subargument of one of these four. Both $v_{\top;*;\bot}$ and $v_{\top;\bot;\bot}$ are counterexamples to the first of these; both $v_{*;\top;\bot}$ and $v_{\bot;\top;\bot}$ to the second; $v_{\top;\bot;\top}$ to the third, and $v_{\bot;\top;\top}$ to the fourth. So each of V_1 and V_2 contains counterexamples to every argument not in A; that is, $A \subseteq \mathcal{A}V_i$ for $i = 1, 2$.

Thus, $\mathcal{A}V_1 = \mathcal{A}V_2 = A$. As neither V_1 nor V_2 is contained in the other, the only way for there to be a least V with $\mathcal{A}V = A$ would be for there to be some $V \subseteq V_1 \cap V_2$ with $\mathcal{A}V = A$. But $V_1 \cap V_2 = \{v_{\top;\bot;\top}, v_{\bot;\top;\top}\}$; this set contains no counterexample to either $[p \rhd r]$ or $[q \rhd r]$. So any $V \subseteq V_1 \cap V_2$ must be such that $\{[p \rhd r], [q \rhd r]\} \subseteq \mathcal{A}(V)$, and so $\mathcal{A}V \neq A$.

□

5.4.3 Reflexive trivaluations

The same example as above suffices to settle the case for reflexive trivaluations via the following Lemma.

Lemma 5. *If A is a reflexive set of arguments and $\mathcal{A}V = A$, then $V \subseteq \mathfrak{V}_3^r$.*

Proof. If $V \not\subseteq \mathfrak{V}_3^r$, then there is some $v \in V$ and $\phi \in \mathcal{L}$ with $v(\phi) = \mathtt{I}$, and so $\mathcal{A}V$ is not reflexive. □

That is, all the tetravaluations we need to consider for any reflexive set of arguments are reflexive trivaluations; the remaining tetravaluations don't get involved. So the above example already gives us a compact reflexive and monotonic set of arguments with no least set of reflexive trivaluations determining it.

[11] For reference, this means that A is the following set of arguments $\{[p \rhd p], [q \rhd q], [r \rhd r], [p, q \rhd p], [p, q \rhd q], [p, r \rhd p], [p, r \rhd r], [q, r \rhd q], [q, r \rhd r], [p, q, r \rhd p], [p, q, r \rhd q], [p, q, r \rhd r], [p, q \rhd r]\}$.

5.4.4 Transitive trivaluations

For transitive trivaluations, however, we need a different example. This is because, even when a set A of arguments is completely transitive, we can still have $A = \mathcal{A}V$ even if $V \not\subseteq \mathfrak{V}^t_3$. So finding a completely transitive set with no least set of tetravaluations determining it is not enough; there might yet be a least set of transitive trivaluations. Moreover, we already know that if we have a set of arguments that is RMT and compact, there will be a least set of transitive trivaluations determining it: since the set is reflexive, it can only be determined by sets of reflexive trivaluations, and the only valuations that are both reflexive trivaluations and transitive trivaluations are the bivaluations, so this case reduces to the bivaluational case, in which we know compactness suffices for a least such set.

What about nonreflexive sets, though? Here, an even simpler example than the last one suffices to show that there are monotonic completely transitive compact sets of Set-Fmla arguments with no least set of valuations determining them. Consider a small language $\mathcal{L} = \{p, q\}$, and the set of arguments $A = \{[p, q \rhd q]\}$. This set is monotonic, completely transitive, and compact.

Theorem 11. *For this A, there is no least set V of transitive trivaluations such that $\mathcal{A}_{SF}V = A$.*

Proof. Let v^x_y be the valuation assigning the value x to p and the value y to q, and consider the following two sets. The first, V_3, is $\{v^{\mathrm{I}}_{\perp}, v^{\perp}_{\mathrm{I}}, v^{\mathrm{I}}_{\top}\}$. The second, V_4, is $\{v^{\top}_{\perp}, v^{\perp}_{\mathrm{I}}, v^{\mathrm{I}}_{\top}\}$. (They differ only in their first member.)[12]

Neither contains a counterexample to $[p, q \rhd q]$, since only $v^{\top}_{\mathrm{I}}$ and $v^{\mathrm{I}}_{\mathrm{I}}$ are such counterexamples. So $\mathcal{A}V_i \subseteq A$ for $i = 3, 4$.

There are seven remaining Set-Fmla arguments in this language, but only three need to be checked (since the other four are subarguments of these): $[p \rhd q]$, $[q \rhd q]$, and $[p, q \rhd p]$. $v^{\mathrm{I}}_{\perp}$ and $v^{\top}_{\perp}$ are both counterexamples to the first; $v^{\perp}_{\mathrm{I}}$ is a counterexample to the second; and $v^{\mathrm{I}}_{\top}$ is a counterexample to the third. So each of V_1 and V_2 contains counterexamples to every argument not in A; that is, $A \subseteq \mathcal{A}(V_i)$ for $i = 3, 4$.

Thus, $\mathcal{A}V_3 = \mathcal{A}V_4 = A$. As neither V_3 nor V_4 is contained in the other, the only way for there to be a least V with $\mathcal{A}V = A$ would be for there to be some $V \subseteq V_3 \cap V_4$ with $\mathcal{A}V = A$. But $V_3 \cap V_4 = \{v^{\perp}_{\mathrm{I}}, v^{\mathrm{I}}_{\top}\}$; this set contains no counterexample to $[p \rhd q]$. So any $V \subseteq V_3 \cap V_4$ must be such that $[p \rhd q] \in \mathcal{A}V$, and so $\mathcal{A}V \neq A$. $\square$

Given the above results, then, it is clear that Malinowski valuations cannot be the least set of valuations consistent with a given set of arguments, as there may not be *any* least set of valuations. The next natural thing to wonder, then, is whether the set Malinowksi valuations are the least set of valuations when such a set exists. As it happens, though, not even this is the case. To see this consider the language $\mathcal{L} = \{p, q\}$ again, and new set of arguments A', where A' is the monotonic closure of the set $\{[\rhd p]\}$ (i.e. $A' = \{[\rhd p], [p \rhd p], [q \rhd p], [p, q \rhd p]\}$). The set of Malinowski valuations for this set of arguments are, letting v^x_y be the valuation assigning x to p and y to q, the following:

---

[12] It's perhaps worth noting that V_3 consists only of Malinowski valuations for A: it is $\{m^{\{p\}}_A, m^{\{q\}}_A, m^{\{p,q\}}_A\}$. (This is how this example was found.) However, it is not $\mathcal{M}A$, since it omits $m^{\emptyset}_A = v^{\perp}_{\perp}$.

$$
\begin{array}{rcl}
m_{A'}^{\emptyset} & = & v_{\bot}^{*} \\
m_{A'}^{\{p\}} & = & v_{\bot}^{\top} \\
m_{A'}^{\{q\}} & = & v_{\overline{\bot}}^{*} \\
m_{A'}^{\{p,q\}} & = & v_{\overline{\bot}}^{\top}
\end{array}
$$

Now by theorem 2 this set of valuations determines A'. As it happens, though, this is not the minimal set of valuations which determines A'. To show this it will be helpful to prove a general result which shows why the Malinowski valuations for a set of arguments are rarely guaranteed to be the minimal set of valuations determining that set of arguments.

Theorem 12. *Suppose that $S \subseteq \mathcal{M}A$ is a set of valuations such that for all $m \in \mathcal{M}A$ there is a $v \in S$ such that $m \sqsubseteq v$. Then $S \in Mod(A)$.*

Proof. First we show that $\mathcal{A}(S) \subseteq A$. Suppose that there is an $a \notin A$. Then by theorem 2 there is an $m \in \mathcal{M}A$ such that $m ※ a$. By the construction of S, though, there is a $v \in S$ such that $m \sqsubseteq v$, so by fact 1 of French and Ripley [201X] it follows that $v ※ a$, as desired.

Second we show that $A \subseteq \mathcal{A}(S)$. If there is a $v \in S$ such that $v ※ a$ for some argument a, then from the fact that $S \subseteq \mathcal{M}A$ it follows that there is an $m \in \mathcal{M}A$ (i.e. v) such that $m ※ a$, and so by theorem 2 it follows that $a \notin A$.

$\square$

To see how this applies in the above case note that $m_{A'}^{\{p,q\}}$ (alias $v_{\overline{\bot}}^{\top}$) is higher in the information ordering than all the other members of $\mathcal{M}A'$, and in this case (being a single valuation) constitutes a minimal set of valuations determining A'. So even in cases where minimal sets of valuations exist, they are likely to not be the Malinowski valuations, as the full set of Malinowski valuations will often carry too much redundant information.

6 Conclusion

The Galois tradition and the Malinowski tradition are related in a number of subtle ways. Each gives a method of determining a set of valuations from a set of arguments, in a way that suffices for an abstract soundness and completeness theorem. The Malinowski tradition in particular has quite a lot of internal texture. What the above results seem to suggest is that in order to get a better grip on the Malinowski valuations the place to look is at how they relate to the information order on valuations.

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Appendix: Least sets of valuations in the Set-Set framework

Recall that, given a set A of arguments, there is a set $Mod(A) = \{V : \mathcal{A}V = A\}$. Consider $Mod(A)$ as ordered by $\subseteq$. No matter whether we're working Set-Set or Set-Fmla, and no matter whether we're considering $\mathfrak{V}_4$, $\mathfrak{V}^t_3$, $\mathfrak{V}^r_3$, or $\mathfrak{V}_2$, $Mod(A)$ has a greatest element: $\mathcal{V}A$. This follows from theorem 1. What we want to answer in this appendix is the following question: under what conditions does it have a least element? In the body of the paper we dealt with the four Set-Fmla cases, and in this appendix we'll look at the remaining four Set-Set cases.

As above, we begin by looking at what happens when we are considering bivaluations, since this is the best-known and best-explored area, and we can largely answer our question by appealing to or adapting existing results. Second, we turn to the general situation involving tetravaluations. We follow up by considering in turn both reflexive, as well as transitive trivaluations.

Bivaluations

It is known (for example [Dunn and Hardegree, 2001, p. 202]) that for any monotonic reflexive completely transitive set of Set-Set arguments A there is exactly one set V of bivaluations such that $\mathcal{A}_{ss}V = A$. That suffices to answer our question for this case: there is always a least such V, since there is always exactly one such V.

Tetravaluations

Things get trickier when we go to the tetravaluational case. Here, there can be multiple distinct Vs with $\mathcal{A}_{ss}V = A$. For example, for any nonempty V, $\mathcal{A}_{ss}V = \mathcal{A}_{ss}(V \cup \{v_*\})$, where v_* is the valuation assigning $*$ to every formula. (This is because v_* is a counterexample only to the empty argument $[\succ]$, and every valuation is a counterexample to this argument.)

We will show that when A is a *compact* monotonic set of Set-Set arguments, then there is a least set V of valuations with $\mathcal{A}_{ss}V = A$. To do this, we first show that there are certain valuations that must be in any V with $\mathcal{A}_{ss}V = A$; then we show that, so long as A is compact, these valuations alone are enough to determine A precisely.

The needed valuations are the *exact counterexamples* to those arguments that are *maximally out* of A. We define each of these notions in what follows, proving needed results along the way. (We earlier defined the notion of *exact counterexample* for SET-FMLA arguments, but here we need it for SET-SET.)

Definition 9. Given a SET-SET argument $a = [\Gamma \succ \Delta]$, its *exact counterexample* v_a is the valuation such that $v_a(\phi) =$

- $\top$ iff $\phi \in \Gamma \setminus \Delta$,
- $\bot$ iff $\phi \in \Delta \setminus \Gamma$,
- I iff $\phi \in \Gamma \cap \Delta$, and
- $*$ iff $\phi \notin \Gamma \cup \Delta$.[13]

Proposition 8. *For any arguments a, b, we have $v_a \divideontimes b$ iff $b \sqsubseteq a$.*[14]

Proposition 9. *For any argument a and any valuation v, we have $v \divideontimes a$ iff $v_a \sqsubseteq v$.*

Proof. Unpacking definitions, in both cases. $\square$

It follows from each of these results that $v_a \divideontimes a$; an argument's exact counterexample is indeed a counterexample. Proposition 8 gives us one sense in which this counterexample is 'exact': it is a counterexample to all and only subsequents of a. Proposition 9 gives us a different sense: it is information-least among counterexamples to a.

Now, to arguments that are maximally out:

Definition 10. An argument is *maximally out* of a set A of arguments iff: it is not in A and any proper superargument of it is in A.

Lemma 6. *If A is a set of SET-SET arguments, and the argument c is maximally out of A, then for any set V of valuations with $\mathcal{A}V = A$ it must be that $v_c \in V$.*

Proof. Take any such c, A, V, to show $v_c \in V$. Since $c \notin A$ and $\mathcal{A}V = A$, there must be some $v \in V$ with $v \divideontimes c$. By proposition 9, $v_c \sqsubseteq v$. Suppose towards a contradiction that $v \neq v_c$. Then there must be some formula ϕ receiving a different value in v than in v_c. Since $v_c \sqsubseteq v$, there are five possibilities:

1. $v_c(\phi) = *$ and $v(\phi) = \top$

[13] An anonymous referee called our attention to the similarity between this definition and the definition of the function called $\flat$ in [Blasio et al., 2017, §5]. These are different functions put to different uses: v_a is a function from the language to values, determined by a particular argument; while $\flat$ is a function from other possible sets of values to our familiar four values, determined by a particular 'B-matrix' (see [Blasio et al., 2017] for definition). We suspect the referee is onto something, and there may be value in pursuing the analogy between formulas and arguments on the one hand, and other value spaces and B-matrices on the other.

[14] In [French and Ripley, 201X], we used proposition 8 as our definition of exact counterexample, and then used the valuations given in definition 9 to show that they always exist. Here, it's convenient to take the reverse approach.

2. $v_c(\phi) = *$ and $v(\phi) = \bot$
3. $v_c(\phi) = *$ and $v(\phi) = \mathrm{I}$
4. $v_c(\phi) = \top$ and $v(\phi) = \mathrm{I}$
5. $v_c(\phi) = \bot$ and $v(\phi) = \mathrm{I}$

But on any of these possibilities, v is a counterexample to some proper superargument of c: in the first, third, and fifth cases, the argument adds ϕ to the conclusions of c, while in the first, second, and fourth, it adds ϕ to the premises. These are indeed proper superarguments: definition 9, plus what we know about $v_c(\phi)$ in each case, suffices for this. But since c is maximally out of A, this superargument is in A, and so $\mathcal{A}V \neq A$, which is a contradiction. Thus, $v = v_c$, and so $v_c \in V$. □

So every $V \in Mod(A)$ must include all the exact counterexamples to those arguments maximally out of A. This matters when A is compact because there are enough arguments maximally out of it:

Proposition 10. *If A is compact, then every argument $a \notin A$ is contained in some argument c that is maximally out of A.*

Proof. Take a compact A and some $a \notin A$. Consider the set $B = \{b \mid a \sqsubseteq b \;\&\; b \notin A\}$ of all superarguments of a that are not in A. Every $\sqsubseteq$-chain in B has an upper bound in B: if the chain is finite, its maximum member will do; and if it is infinite, its sequent join will do. (Note that Compactness is needed at this step to ensure that these joins are $\notin A$). But then by Zorn's lemma, B has a maximal element c. Since $c \in B$, we know $a \sqsubseteq c$ and $c \notin A$.

To show that c is maximally out of A, it remains only to show that any proper superargument d of c is in A. But if there were some $c \sqsubsetneq d$ where $d \notin A$, then d would have to have been in B, and so c would not be maximal in B after all. □

This is enough now for the theorem.

Theorem 13. *If A is a compact monotonic set of* Set-Set *arguments, then there is a least $V \in Mod(A)$.*

Proof. By lemma 6, any $V \in Mod(A)$ must be such that $V_0 = \{v_c \mid c \text{ is maximally out of } A\} \subseteq V$. So if $\mathcal{A}V_0 = A$, then $V_0 \in Mod(A)$ and we're done. Showing this has two phases: that $\mathcal{A}V_0 \subseteq A$ and that $A \subseteq \mathcal{A}V_0$.

First, that $\mathcal{A}V_0 \subseteq A$. Take any $a \notin A$. By proposition 10, there is some c maximally out of A with $a \sqsubseteq c$. Since c is maximally out of A we have $v_c \in V_0$, and by one direction of proposition 8 we have v_c ※ a. So $a \notin \mathcal{A}V_0$.

Second, that $A \subseteq \mathcal{A}V_0$. Take any $a \notin \mathcal{A}V_0$; this has some counterexample $v_c \in V_0$. By the other direction of proposition 8, $a \sqsubseteq c$. But since c is maximally out of A, it is at least out of A; and since A is monotonic, $a \notin A$. □

Reflexive trivaluations

If A is a reflexive set of arguments, then the general story carries over immediately, owing to lemma 5. So when A is reflexive and compact, the least set of tetravaluations determining A (which exists by theorem 13) is a set of reflexive trivaluations by lemma 5, and thus a least set of reflexive trivaluations determining A.

Transitive trivaluations

For transitive trivaluations, the reasoning is not so immediate, because there is no result analogous to lemma 5 available for complete transitivity and transitive trivaluations.[15] But we can still make our way to the corresponding result; restricting our attention to completely transitive sets of arguments gives us extra tools to work with.

Proposition 11. *If A is monotonic and completely transitive, and $[\Gamma \succ \Delta]$ is maximally out of A, then $\Gamma \cup \Delta = \mathcal{L}$.*

Proof. Suppose A is monotonic and completely transitive, and $[\Gamma \succ \Delta]$ is maximally out of A, but that $\Gamma \cup \Delta \neq \mathcal{L}$. Then there must be some $\phi \in \mathcal{L}$ with $\phi \notin \Gamma \cup \Delta$. So both $[\Gamma \succ \Delta, \phi]$ and $[\phi, \Gamma \succ \Delta]$ are proper superarguments of $[\Gamma \succ \Delta]$. Since $[\Gamma \succ \Delta]$ is maximally out of A, both of these superarguments must be in A. But then since A is completely transitive, $[\Gamma \succ \Delta] \in A$; contradiction.[16] □

This is now enough to proceed.

Theorem 14. *If A is a monotonic, completely transitive, and compact set of SET-SET arguments, then there is a least set V of transitive trivaluations determining it.*

Proof. As in the proof of theorem 13, the desired set is the set of exact counterexamples to those arguments maximally out of A. As we have seen in that proof, this set is the least set of tetravaluations determining A. So as long as it is itself a set of transitive trivaluations, we're done. By proposition 11, every $[\Gamma \succ \Delta]$ maximally out of A is such that $\Gamma \cup \Delta = \mathcal{L}$. Consulting definition 9 reveals that the exact counterexample to any such argument cannot use the value $*$, and so is a transitive trivaluation. □

[15] Indeed, $\mathcal{V}_4 A \not\subseteq \mathfrak{V}_3^t$ unless $\mathcal{V}_4 A$ is empty, but so long as A is monotonic we have $\mathcal{A}\mathcal{V}_4 A = A$.

[16] The full strength of complete transitivity wasn't needed here; the weaker property [Shoesmith and Smiley, 1978] calls 'cut for formulas' is enough. But since we're only going to apply proposition 11 in cases where we're also assuming compactness, and since in the presence of compactness cut for formulas suffices for complete transitivity, it wouldn't be worth stating the slightly stronger formulation of proposition 11 that its proof makes possible.