The Constructive Logic of Paradox: Paraconsistency on the Woodruff Plan

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Abstract

In this paper we investigate the constructive companion of the logic of paradox. After presenting the logic semantically, and examining how it relates to both intuitionistic logic and the logic of paradox, we prove completeness for two semantically conservative extensions of the constructive logic of paradox—a non-paraconsistent one with a 'true-only' predicate, and a non-constructive one with a De Morgan negation.

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1 Introduction

In his paper "On Constructive Nonsense Logic" Woodruff states his goal as the following, where D is the logic of nonsense from [10]:

> Our goal will be to produce a calculus which stands to D as intuitionistic does to classical logic, and at the same time stands to intuitionistic logic as D does to classical logic. [14, p. 195]

The logic that Woodruff goes on to characterize combines features of the semantics of intuitionistic logic—its frames being pre-orders—with features of 'dual valuation' semantics for logics of nonsense.[1] While the semantics for the logic we will investigate

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[1] Arenhart & Omori argue in [1] that because of this dual valuational semantics, among many other reasons, Woodruff's logic is not really a logic of nonsense. The interested reader should consult [6, p. 358–362] for a discussion of Woodruff's and some neighboring systems of constructive nonclassical logic.

below exhibits similar features (being built on pre-orders, but combined with the 'dual valuation' treatment of many-valued logics implicit in [4]), what we take as our starting point for the present paper is the idea, implicit in the above quote, of understanding the constructivisation of a logic L as involving it being appropriately similar to both L and to IL. We can think of this as a 'relational' dimension of similarity between logics, alluded to by the claim that DI should be appropriately related to both D and IL, to which we might plausibly wish to add a more 'intrinsic' dimension of similarity—in the above case it seems plausible that the constructive nonsense logic DI ought to be both constructive and a logic of nonsense.

In this paper, we set out to apply what we will call 'The Woodruff Plan' for building constructive nonclassical logics in a new setting, paraconsistent logic. In particular, we are interested in characterizing and understanding the constructive analogue of the three-valued logic of paradox LP of [9].[2]

To do this we will need to get clear on how to understand the following two claims:

- L stands to LP as IL stands to CL.
- L stands to IL as LP stands to CL.

We will give a relatively coarse grained syntactic notion of 'standing to' here, understanding the claim that L to stand to LP as IL stands to CL as requiring that there be some collection of sequents S such that the extension of L by S results in LP and the extension of IL by S results in CL. We will then combine this relational notion of 'standing to' with the restriction that our logic L must be constructive/ paraconsistent to get our notion of a *constructive companion* and a *paraconsistent companion*.

Definition 1. S^{c} is a *constructive companion* of S iff (i) S^{c} has the disjunction property, and (ii) there is an S-valid sequent s which, when added to IL gives CL, and which when added to S^{c} gives S.

Definition 2. S^{p} is a *paraconsistent companion* of S iff (i) in S^{p} there is no nonempty finite set of sentences Γ such that $[\Gamma \succ\ ]$ is valid, and (ii) the result of adding explosion (i.e., $[\Gamma, A, \neg A \succ \Delta]$) to S^{p} gives S.

Note that here we are understanding paraconsistency in a language neutral way requiring that for a logic to be paraconsistent there can be no nonempty finite set

[2]Our main interest in this logic, although not on display otherwise in the present paper, arises out of an interest in the constructive versions of the non-transitive logic ST of [4], it being well known that this logic is in some sense translationally equivalent to LP, as pointed out in [2] and [5].

of formulas which behave like $\{A, \neg A\}$ do in classical logic.[3]

Our goal in the present paper will be to investigate a particular logic cLP, which we will show is a constructive companion of LP and a paraconsistent companion of IL. For the sake of simplicity (and ease of comparison with LP) we will be considering only languages with the primitive connectives $\wedge$, $\vee$ and $\neg$. The omission of the conditional is purely for ease of exposition and its addition would not repair any of the difficulties which we point out along the way.

The plan for the remainder of the paper is as follows. In §2 we introduce the constructive logic of paradox model-theoretically and show that it is a constructive companion of LP and a paraconsistent companion of IL. We will then look at two conservative extensions of this logic for which we will prove soundness and completeness results, in §3 by the addition of a 'just true' operator, and in §4 by the addition of a De Morgan negation.

Some preliminaries before we begin. Our language $\mathcal{L}$ consists of formulas constructed out of a countable stock $p_1, \ldots, p_n, \ldots$, of propositional variables, the first three of which we will usually abbreviate as p, q, r, using the connectives $\wedge$, $\vee$ and $\neg$. Capital Roman letters are schematic for formulas of our language; capital Greek letters (that are not also Roman letters) are schematic for (typically finite) multisets of formulas. Throughout we work in the SET-SET framework, with logics being identified with generalized consequence relations $\vdash \subseteq \wp(\mathcal{L}) \times \wp(\mathcal{L})$. Accordingly, throughout, a *sequent* is an ordered pair $\langle \Gamma, \Delta \rangle$ of finite multisets of formulas which we will write as $[\Gamma \succ \Delta]$.

2 Semantics for the Constructive Logic of Paradox

Our models are a particular kind of intuitionistic Kripke models with a pair of valuations, investigated initially in [8], and more recently in [7]. Formally speaking, a model is a tuple $\langle W, \leq, V_s, V_t \rangle$ where W is a nonempty set, $\leq$ is a reflexive and transitive relation of accessibility on W, and V_s and V_t are functions from propositional atoms to subsets of W which satisfy the following three conditions for all $w, v \in W$ and propositional variables p_i:

- PERSISTENCE-S If $w \in V_s(p_i)$ and $w \leq v$, then $v \in V_s(p_i)$.
- PERSISTENCE-T If $w \in V_t(p_i)$ and $w \leq v$, then $v \in V_t(p_i)$.
- CONTAINMENT $V_s(p_i) \subseteq V_t(p_i)$.

[3] That is to say, we require that a paraconsistent logic not satisfy the principle *sEQC* of 'set-based explosion', the strongest explosion principle surveyed in [3].

Truth at a point in a model is defined as follows.

1. $\mathcal{M}, w \Vdash_s p_i$ iff $w \in V_s(p_i)$
2. $\mathcal{M}, w \Vdash_t p_i$ iff $w \in V_t(p_i)$
3. $\mathcal{M}, w \Vdash_s A \wedge B$ iff $\mathcal{M}, w \Vdash_s A$ and $\mathcal{M}, w \Vdash_s B$
4. $\mathcal{M}, w \Vdash_t A \wedge B$ iff $\mathcal{M}, w \Vdash_t A$ and $\mathcal{M}, w \Vdash_t B$
5. $\mathcal{M}, w \Vdash_s A \vee B$ iff $\mathcal{M}, w \Vdash_s A$ or $\mathcal{M}, w \Vdash_s B$
6. $\mathcal{M}, w \Vdash_t A \vee B$ iff $\mathcal{M}, w \Vdash_t A$ or $\mathcal{M}, w \Vdash_t B$
7. $\mathcal{M}, w \Vdash_s \neg A$ iff for all $w \leq w'$ we have $\mathcal{M}, w' \nVdash_t A$
8. $\mathcal{M}, w \Vdash_t \neg A$ iff for all $w \leq w'$ we have $\mathcal{M}, w' \nVdash_s A$

We will say that a sequent $[\Gamma \succ \Delta]$ *holds in a model* $\mathcal{M}$ iff for all worlds w we have either $\mathcal{M}, w \nVdash_t \gamma$ for some $\gamma \in \Gamma$, or $\mathcal{M}, w \Vdash_t \delta$ for some $\delta \in \Delta$. A sequent $[\Gamma \succ \Delta]$ is *valid on a frame* $\langle W, \leq \rangle$ iff it holds in every model $\langle W, \leq, V_s, V_t \rangle$ on that frame. Finally, a sequent is *valid in a class of models* M iff it holds in every $\mathcal{M} \in \mathsf{M}$, and is *valid on a class of frames* F iff it is valid on every frame $\mathfrak{F} \in \mathsf{F}$. Finally, let $\Gamma \vdash_{\mathsf{cLP}} \Delta$ iff $[\Gamma \succ \Delta]$ is valid on all frames.

Just like with the Kripke semantics for intuitionistic logic, our semantic clauses are such that by stipulating the conditions PERSISTENCE-T, PERSISTENCE-S, and CONTAINMENT for propositional variables we are able to show that these conditions apply to all formulas in the language.

Lemma 1 (Persistence). *For all models $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$, formulas A and points $w, v \in W$ we have:*

1. *If $\mathcal{M}, w \Vdash_s A$ and $w \leq v$, then $\mathcal{M}, v \Vdash_s A$.*
2. *If $\mathcal{M}, w \Vdash_t A$ and $w \leq v$, then $\mathcal{M}, v \Vdash_t A$.*

Lemma 2 (Containment). *For all models $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$, formulas A and points $w, v \in W$ we have that if $\mathcal{M}, w \Vdash_s A$, then $\mathcal{M}, w \Vdash_t A$.*

Moreover, witnessing the fact that cLP is constructive, we can show that it satisfies the Disjunction property.

Lemma 3 (Disjunction Property). *If $\vdash_{\mathsf{cLP}} A \vee B$, then either $\vdash_{\mathsf{cLP}} A$ or $\vdash_{\mathsf{cLP}} B$.*

Proof. This follows by the standard model-theoretic argument. If both sequents are invalid then there are models $\mathcal{M}_A$ and $\mathcal{M}_B$ where they fail to hold. Construct a new model $\mathcal{M}$ by taking the disjoint union of these two models and adding a new point w from which every point in each model is accessible. By T-PERSISTENCE we cannot have either $\mathcal{M}, w \Vdash_t A$ or $\mathcal{M}, w \Vdash_t B$, and so we have $\mathcal{M}, w \nVdash_t A \vee B$, as desired. □

What we will show in the next two subsections is that this logic is indeed both a constructive counterpart of LP, and a paraconsistent counterpart of IL.

2.1 Constructive Counterpart

It is very easy to show that, just as classical logic is determined by the class of all intuitionistic Kripke frames where the accessibility relation is the identity relation, so too is the logic of paradox determined by the class of all frames of the above kind. Moreover, these frames are precisely those frames which validate the law of excluded middle.

Theorem 1. *All instances of the law of excluded middle are valid on a frame iff its accessibility relation is the identity relation.*

Proof. Let $\langle W, = \rangle$ be a frame, and suppose $\mathcal{M} = \langle W, =, V_s, V_t \rangle$ is a model on that frame. Suppose that $\mathcal{M}, w \nVdash_t A$. Then by CONTAINMENT we have $\mathcal{M}, w \nVdash_s A$. So at all worlds v such that $w = v$ we have $\mathcal{M}, v \nVdash_s A$, and so $\mathcal{M}, w \Vdash_t \neg A$, as desired.

Let $\langle W, \leq \rangle$ be a frame where $w \leq v$ and $w \neq v$. Let $V_s(p) = V_t(p) = \{u : v \leq u\}$. As $w \leq v$ and $v \Vdash_s p$ it follows that $w \nVdash_t \neg p$. It is clear that $w \notin V_t(p)$ and so $w \nVdash_t p$ and so the instance $[\,\succ p \vee \neg p]$ fails to be valid on $\langle W, \leq \rangle$. □

To show that the logic determined by the class Id of frames whose accessibility relation is the identity relation is LP it will be helpful to specify LP-validity and counterexamples in the following way.

Definition 3. A *Strong-Kleene valuation* is a function $v : \mathcal{L} \to \{1, \frac{1}{2}, 0\}$ where:

- $v(\neg A) = 1 - v(A)$
- $v(A \wedge B) = \min(v(A), v(B))$
- $v(A \vee B) = \max(v(A), v(B))$

A Strong Kleene valuation v is an LP-*counterexample* to a sequent $[\Gamma \succ \Delta]$ iff $v[\Gamma] \subseteq \{1, \frac{1}{2}\}$ and $v[\Delta] = 0$. A sequent is LP-*valid* iff it has no LP-counterexamples.

The following results are analogues of the familiar results connecting up the endpoints of intuitionistic Kripke frames with boolean valuations.

Theorem 2. *Suppose that $\langle W, \leq, V_s, V_t\rangle$ is a model where $\leq$ is identity. Consider the following trivaluation v_w*

$$v_w(A) = \begin{cases} 1 & \text{if } w \Vdash_s A \\ \frac{1}{2} & \text{if } w \Vdash_t A \text{ and } w \nVdash_s A \\ 0 & \text{if } w \nVdash_t A \end{cases}$$

Then v_w is a Strong Kleene valuation such that $v_w(A) \in \{1, \frac{1}{2}\}$ iff $\mathcal{M}, w \Vdash_t A$.

Proof. The only difficulty in the proof is verifying that v_w is a Strong-Kleene valuation. We show the case of negation, the others being similar.

$v_w(B) = 1$: $v_w(B) = 1$ iff $w \Vdash_s B$ iff $w \nVdash_t \neg B$ (as $\leq$ is identity) iff $v_w(\neg B) = 0$, as desired.

$v_w(B) = \frac{1}{2}$: $v_w(B) = \frac{1}{2}$ iff $w \Vdash_t A$ and $w \nVdash_s A$ iff $w \nVdash_s \neg A$ and $w \Vdash_t \neg A$ (as $\leq$ is identity) iff $v_w(\neg B) = \frac{1}{2}$, as desired.

$v_w(B) = 0$: $v_w(B) = 0$ iff $w \nVdash_t B$ iff $w \Vdash_s \neg B$ (as $\leq$ is identity) iff $v_w(\neg B) = 1$, as desired. □

Theorem 3. *Suppose that v is a Strong Kleene valuation. Let $\mathcal{M}_v = \langle\{w\}, =, V_s^v, V_t^v\rangle$ be the model where $w \in V_s^v(p_i)$ iff $v(p_i) = 1$ and $w \in V_t^v(p_i)$ iff $v(p_i) \in \{1, \frac{1}{2}\}$. Then $v(A) \in \{1, \frac{1}{2}\}$ iff $\mathcal{M}_v, w \Vdash_t A$.*

Proof. We prove something stronger, namely, that the following equivalences hold:

1. $v(B) = 1$ iff $\mathcal{M}_v, w \Vdash_s B$;
2. $v(B) = \frac{1}{2}$ iff $\mathcal{M}_v, w \nVdash_s B$ and $\mathcal{M}_v, w \Vdash_t B$;
3. $v(B) = 0$ iff $\mathcal{M}_v, w \nVdash_t B$.

We proceed by induction on the complexity of A. The basis case is trivial. Suppose then, that for all formulas of complexity less than n the above three equivalences hold. We treat the case in the induction step for conjunction.

$v(B \wedge C) = 1$: $v(B \wedge C) = 1$ iff $v(B) = 1$ and $v(C) = 1$ iff by the induction hypothesis $\mathcal{M}_v, w \Vdash_s B$ and $\mathcal{M}_v, w \Vdash_s C$ iff $\mathcal{M}_v, w \Vdash_s B \wedge C$, as desired.

$v(B \wedge C) = \frac{1}{2}$: $v(B \wedge C) = \frac{1}{2}$ iff either (i) $v(B) = \frac{1}{2}$ or $v(C) = \frac{1}{2}$ and (ii) $v(B) \in \{1, \frac{1}{2}\}$ and (iii) $v(C) \in \{1, \frac{1}{2}\}$. Suppose wlog that $v(B) \in \frac{1}{2}$. Then by the induction hypothesis, $\mathcal{M}_v, w \not\Vdash_s B$ and so $\mathcal{M}_v, w \not\Vdash_s B \wedge C$. Further, from (iii) by the induction hypothesis, we have $\mathcal{M}_v, w \Vdash_t C$ and so by (i) and the induction hypothesis $\mathcal{M}_v, w \Vdash_t B \wedge C$.

$v(B \wedge C) = 0$: $v(B \wedge C) = 0$ iff $v(B) = 0$ or $v(C) = 0$ iff by the induction hypothesis $\mathcal{M}_v, w \not\Vdash_t B$ or $\mathcal{M}_v, w \not\Vdash_t C$ iff $\mathcal{M}_v, w \not\Vdash_t B \wedge C$, as desired. □

We can use this result to easily show that cLP is paraconsistent in the strong sense of there being no set of formulas Γ such that $[\Gamma \succ\ ]$.

Lemma 4 (Strong Paraconsistency)**.** *For no set of formulas Γ, do we have $\Gamma \vdash_{\mathsf{cLP}}$.*

Proof. The valuation v_* where $v_*(p_i) = \frac{1}{2}$ for all propositional variables p_i is such that $v_*(A) = \frac{1}{2}$ for all formulas A. So by Theorem 3, it follows that $\mathcal{M}_v, w \Vdash_t A$ for all formulas A, and so for all formulas $C \in \Gamma$. □

Theorem 4. *A sequent $[\Gamma \succ \Delta]$ is valid in all models on frames in* Id *iff $[\Gamma \succ \Delta]$ is* LP *valid.*

Proof. Suppose that $[\Gamma \succ \Delta]$ is not LP-valid. Then there is some Strong Kleene valuation v such that $v[\Gamma] \subseteq \{1, \frac{1}{2}\}$ and $v[\Delta] = 0$. By Theorem 3, it follows that $\mathcal{M}_v, w \Vdash_t C$ for all $C \in \Gamma$ and $\mathcal{M}_v, w \not\Vdash_t D$ for any $D \in \Delta$, and so $[\Gamma \succ \Delta]$ is not valid on Id.

Suppose then, that $[\Gamma \succ \Delta]$ is not valid on Id. Then there is a model $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$ in Id where at some $w \in W$, we have $\mathcal{M}, w \Vdash_t C$ for all $C \in \Gamma$ and $\mathcal{M}, w \not\Vdash_t D$ for all $D \in \Delta$. By Theorem 2 it follows that $v_w[\Gamma] \subseteq \{1, \frac{1}{2}\}$, and $v_w[\Delta] = 0$ and so $[\Gamma \succ \Delta]$ is invalid in LP. □

From the above we can see that cLP is a constructive companion of LP, as the logic determined by the class of frames which validate the law of excluded middle is LP.

2.2 Paraconsistent Counterpart

Unlike the case for the law of excluded middle, there is no class of *frames* which validates explosion. In other words the logic we get by extending cLP by all instances of $[\Gamma, A, \neg A \succ \Delta]$ is Kripke-incomplete.[4]

[4]A similar result can be seen to hold for the condition X^+ in [14, p. 203] that meaningfulness is effectively decidable—there the proof essentially relies on the fact that for every frame there is a model on that frame which does not have a strong valuation.

Theorem 5. *Let $\langle W, \leq \rangle$ be a frame. Then there are V_s and V_t such that some instance of $[\Gamma, A, \neg A \succ \Delta]$ fails to hold in $\langle W, \leq, V_s, V_t \rangle$.*

Proof. In particular, we will show that $[p, \neg p \succ q]$ fails to hold in $\langle W, \leq, V_s, V_t \rangle$. Let w be a world in W. Then let $V_t(p) = \{u : w \leq u\}$ (so $w \Vdash_t p$), $V_s(p) = \emptyset$ (so in particular, for all $w \leq v$ we have $v \nVdash_s p$ and so $w \Vdash_t \neg p$), and let $V_t(q) = V_s(q) = \emptyset$ (and so $w \nVdash_t q$). Then the result follows. $\square$

So there is no condition on *frames* which we can impose which will result in all models on frames which meet that condition validating explosion. But we can impose a condition directly on *models*. In particular, we will show that all instances of explosion hold in a model exactly when that model meets a condition we call *delayed converse containment* or DCC.

Theorem 6. *If all instances of $[\Gamma, A, \neg A \succ \Delta]$ hold in $\mathcal{M}$, then $\mathcal{M}$ satisfies*

$$\text{(DCC)} \qquad \forall w \forall p_i (w \in V_t(p_i) \Rightarrow \exists v \geq w . v \in V_s(p_i)).$$

Proof. We prove the contrapositive. Suppose that the above condition is false, that is we have a model $\mathcal{M}$ where for some $w \in W$ we have:

$$w \in V_t(p_i) \text{ and } \forall v \geq w . v \notin V_s(p_i).$$

From this it follows directly that we have $w \Vdash_t p$, and as for all $w \leq v$ we have $v \nVdash_s p$ also $w \Vdash_t \neg p$, and so it is not the case that $[p, \neg p \succ\ ]$ holds in $\mathcal{M}$, and so not all instances of $[\Gamma, A, \neg A \succ \Delta]$ hold in $\mathcal{M}$. $\square$

As with containment, we can also show that models which satisfy delayed converse containment for propositional variables also satisfy it for all formulas.

Theorem 7. *Suppose that $\mathcal{M}$ satisfies (DCC). Then for all formulas A and worlds w we have:*

$$w \Vdash_t A \;\Rightarrow\; \exists v \geq w . v \Vdash_s A.$$

Proof. By induction on the complexity of A. Basis is trivial.

Conjunction For the conjunction case, suppose that $w \Vdash_t A \wedge B$. Then $w \Vdash_t A$ and $w \Vdash_t B$. So there are worlds u, v such that $w \leq u$ and $w \leq v$ where $u \Vdash_s A$ and $v \Vdash_s B$. If $u \leq v$ or $v \leq u$ then at the later one we will have $A \wedge B$ s-verified. If not then note by persistence that at u we have $u \Vdash_t B$ and so (again by IH) $x \Vdash_s B$ for some $u \leq x$, and so $x \Vdash_s A \wedge B$.

Disjunction For disjunction suppose that $w \Vdash_t A \vee B$. Then wlog suppose $w \Vdash_t A$. Then by IH we have that $w \leq u$ and $u \Vdash_s A$ and so $u \Vdash_s A \vee B$.

Negation Suppose $w \Vdash_t \neg A$. Then we have for all $u \geq w$ that $u \nVdash_s A$. Now let v be an arbitrary world where $w \leq v$. If $v \Vdash_t A$ then by the IH we must have some $v \leq x$ such that $x \Vdash_s A$. By the transitivity of accessibility relation, this means that we must have $w \leq x$ and so $x \nVdash_s A$, giving us a contradiction. So it follows that all the worlds $u \geq w$ are such that $u \nVdash_t A$ and so $w \Vdash_s \neg A$. $\square$

One important consequence of this condition is that we can recover the usual intuitionistic truth condition for negation.

Proposition 1. *If $\mathcal{M}$ satisfies (DCC), then we have*

$$w \Vdash_t \neg A \;\Leftrightarrow\; \forall u \geq w. u \nVdash_t A.$$

Proof. Suppose that $\mathcal{M}$ satisfies condition (DCC). Then by Theorem 7, we have that for all formulas A and worlds w, if $w \Vdash_t A$ then there is a v such that $v \geq w$ and $v \Vdash_s A$, and so contraposing this we have

$$\forall v \geq w. v \nVdash_s A \;\Rightarrow\; w \nVdash_t A$$

Now suppose that $w \Vdash_t \neg A$. Then by our truth conditions, it follows that for all $v \geq w$ we have $v \nVdash_s A$. Now suppose that u is a point such that $u \geq w$. Then by the transitivity of $\leq$, we have that all $u' \geq u$ are such that $u' \nVdash_s A$, and so by the above inset condition $u \nVdash_t A$. But u was arbitrary, so we have that $\forall u \geq w. u \nVdash_t A$, as desired.

For the other direction, suppose that we have that for all $u \geq w$, $u \nVdash_t A$. Then by Lemma 2, we have $u \nVdash_s A$, and so by our truth conditions $w \Vdash_t \neg A$, as desired. $\square$

Corollary 5. *If $\mathcal{M}$ satisfies (DCC), then all instances of $[\Gamma, A, \neg A \succ \Delta]$ hold in $\mathcal{M}$.*

Theorem 8 (Paraconsistent Companion)**.** *A sequent $[\Gamma \succ \Delta]$ is valid in the class of all models satisfying DCC iff $\Gamma \vdash_{\mathsf{IL}} \Delta$.*

Proof. Given Proposition 1, it follows that any model $\langle W, \leq, V_s, V_t \rangle$ which satisfies DCC is modally equivalent, in the sense of validating the same formulas, (w.r.t. t-satisfaction) to the intuitionistic Kripke model $\langle W, \leq, V_t \rangle$. As a result if a sequent fails to hold in a given DCC model, it also fails to hold in the corresponding intuitionistic Kripke model. For the converse, if a sequent fails in some intuitionistic Kripke model $\langle W, \leq, V \rangle$ the aforementioned result tells us that the same sequent fails to hold in the model $\langle W, \leq, V, V \rangle$. $\square$

From the above we can see that cLP is a paraconsistent companion of IL, understanding the result of extending cLP by some sequent s as involving taking the logic determined by the class of models which validate s.

2.3 Characterizing cLP

It is a routine exercise to see that the following sequents are valid and invalid according to the above semantics.

VALID	INVALID
$[A \succ \neg\neg A]$	$[\neg\neg A \succ A]$
$[\neg A \vee \neg B \succ \neg(A \wedge B)]$	$[\neg(A \wedge B) \succ \neg A \vee \neg B]$
$[\neg A \wedge \neg B \succ \neg(A \vee B)]$	$[A, \neg A \succ B]$
$[\neg(A \vee B) \succ \neg A \wedge \neg B]$	$[\ \succ A \vee \neg A]$
$[\ \succ \neg(A \wedge \neg A)]$	
$[\neg\neg\neg A \succ \neg A]$	

Looking at this one might reasonably wonder whether cLP is just the intersection of the generalized consequence relations for LP and IL. As it turns out, this is not the case, as the sequent $[p, \neg p \succ q, \neg q]$ is valid in both of these logics, but is not valid in cLP . To see this consider the model $\langle \{0,1\}, \leq, V_s, V_t \rangle$ where $V_t(p) = \{0,1\}$, $V_s(p) = \emptyset$, $V_t(q) = V_s(q) = \{1\}$ and $\leq$ is the reflexive closure of $0 < 1$. In this model, $0 \Vdash_t p$, and as $i \nVdash_s p$ for $i \in \{0,1\}$ we have $0 \Vdash_t \neg p$. But as we have $1 \Vdash_s q$ we have $0 \nVdash_t \neg q$ and also $0 \nVdash_t q$, so the sequent fails to hold at 0.

So cLP is not the intersection of the logic of paradox and intuitionistic logic, although as we have seen above, it is strongly related to them. Unfortunately, we have been unable, in this signature, to give a proof-theoretic characterization of this logic. We can get an indirect grip on cLP proof-theoretically though, through the (conservative) addition of additional connectives, as we will see in the following sections. Each of these extensions repair the expressive deficit we face in proving completeness in distinct ways. Unfortunately, each comes at a cost of either constructivity or paraconsistency.

3 cLP with a Strict Truth Operator: Constructive, but not Paraconsistent

In this section we will consider a signature similar to that considered by Woodruff in [13] (and [14]). The base signature used therein includes not just the connectives

$\{\wedge, \vee, \neg, \rightarrow\}$ but also two additional operators T and $*$ with essentially the following semantic clauses [13, p. 94f]:

$$\mathcal{M}, w \Vdash_s \mathsf{T}A \text{ iff } \mathcal{M}, w \Vdash_s A \qquad \mathcal{M}, w \Vdash_t \mathsf{T}A \text{ iff } \mathcal{M}, w \Vdash_s A$$

$$\mathcal{M}, w \Vdash_s *A \text{ iff } \mathcal{M}, w \Vdash_t A \qquad \mathcal{M}, w \Vdash_t *A \text{ iff } \mathcal{M}, w \Vdash_t A$$

Woodruff refers to these two operators as the 'unconditional' and 'hedged' truth operators. Given its behavior in the present semantics, we will prefer to refer to T as the 'strict truth' operator—roughly speaking the 'unconditional' truth-operator reports that the sentence following it is true, while the 'hedged' truth operator merely reports that the sentence following is not false.

Let $\mathcal{L}^{\mathsf{T}}$ be the language, which results from extending $\mathcal{L}$ by the addition of the strict truth operator T. We will say that a sequent $[\Gamma \succ \Delta]$ of $\mathcal{L}^{\mathsf{T}}$ is valid in $\mathsf{cLP}^{\mathsf{T}}$ iff it is valid on all frames, sometimes writing this as $\Gamma \vdash_{\mathsf{cLP}^{\mathsf{T}}} \Delta$. As it happens extending our language with just this operator allows us to give a sound and complete sequent calculus for the logic $\mathsf{cLP}^{\mathsf{T}}$. The proof system $\mathcal{SCLP}^{\mathsf{T}}$consists of all the rules and basic sequents in Figure 1.

It is a routine matter to show that $\mathcal{SCLP}^{\mathsf{T}}$is sound w.r.t. our semantics enriched with the clauses for the T-operator.

Theorem 9 (Soundness for $\mathcal{SCLP}^{\mathsf{T}}$)**.** *If a sequent* $[\Gamma \succ \Delta]$ *is derivable in* $\mathcal{SCLP}^{\mathsf{T}}$*, then it is valid.*

Proof. By induction on the length of derivations. For the basis case it is easy to see that $[Id]$ is valid. For the case of $[\mathsf{T}ax]$ suppose that $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$ is a model and for some $w \in W$ we have $\mathcal{M}, w \Vdash_t \mathsf{T}A$. Then it follows that $\mathcal{M}, w \Vdash_s A$ and so by Lemma 2, that $\mathcal{M}, w \Vdash_t A$, as desired.

In the induction step, we treat $[\neg R]$, the other cases either being routine or sufficiently similar. We show that if the conclusion sequent fails to hold in a model then so does the premise sequent. Suppose that $[\Gamma \succ \neg A]$ fails to hold at $w \in W$. Then we have $\mathcal{M}, w \Vdash_t C$ for all $C \in \Gamma$ and $\mathcal{M}, w \nVdash_t \neg A$. So there is some $w \leq v$ such that $\mathcal{M}, v \Vdash_s A$ and so $\mathcal{M}, w \Vdash_t \mathsf{T}A$ as desired. □

For completeness, we will use a relatively straightforward canonical model proof, the worlds in our canonical model being suitably saturated pairs of sets of formulas.

Definition 4 (T-Saturation)**.** Say that a pair of sets of formulas (l, r) is T-*saturated* iff $[l \succ r]$ is unprovable in $\mathcal{SCLP}^{\mathsf{T}}$and we have the following:

1. If $A_1 \wedge A_2 \in l$, then $A_1 \in l$ and $A_2 \in l$.

Initial Sequents

$$\frac{}{[\Gamma, A \succ A, \Delta]}\ [Id] \qquad \frac{}{[\Gamma, \mathsf{T}A \succ \Delta, A]}\ [\mathsf{T}ax]$$

Structural Rules

$$\frac{[\Gamma \succ \Delta, A] \quad [A, \Gamma' \succ \Delta']}{[\Gamma, \Gamma' \succ \Delta, \Delta']}\ [Cut]$$

$$\frac{[\Gamma \succ \Delta]}{[A, \Gamma \succ \Delta]}\ [KL] \qquad \frac{[\Gamma \succ \Delta]}{[\Gamma \succ \Delta, A]}\ [KR] \qquad \frac{[A, A, \Gamma \succ \Delta]}{[A, \Gamma \succ \Delta]}\ [WL] \qquad \frac{[\Gamma \succ \Delta, A, A]}{[\Gamma \succ \Delta, A]}\ [WR]$$

Operational Rules

$$\frac{[\Gamma, A, B \succ \Delta]}{[\Gamma, A \wedge B \succ \Delta]}\ [\wedge L] \qquad \frac{[\Gamma \succ \Delta, A] \quad [\Gamma' \succ \Delta', B]}{[\Gamma, \Gamma' \succ \Delta, \Delta', A \wedge B]}\ [\wedge R]$$

$$\frac{[\Gamma, A \succ \Delta] \quad [\Gamma', B \succ \Delta']}{[\Gamma, \Gamma', A \vee B \succ \Delta, \Delta']}\ [\vee L] \qquad \frac{[\Gamma \succ \Delta, A, B]}{[\Gamma \succ \Delta, A \vee B]}\ [\vee R]$$

T-Interaction Rules

$$\frac{[\Gamma \succ \Delta, \mathsf{T}A] \quad [\Gamma \succ \Delta, \mathsf{T}B]}{[\Gamma \succ \Delta, \mathsf{T}(A \wedge B)]}\ [\wedge R\mathsf{T}] \qquad \frac{[\Gamma, \mathsf{T}A, \mathsf{T}B \succ \Delta]}{[\Gamma, \mathsf{T}(A \wedge B) \succ \Delta]}\ [\wedge L\mathsf{T}]$$

$$\frac{[\Gamma, \mathsf{T}A \succ \Delta] \quad [\Gamma, \mathsf{T}B \succ \Delta]}{[\Gamma, \mathsf{T}(A \vee B) \succ \Delta]}\ [\vee L\mathsf{T}] \qquad \frac{[\Gamma \succ \Delta, \mathsf{T}A, \mathsf{T}B]}{[\Gamma \succ \Delta, \mathsf{T}(A \vee B)]}\ [\vee R\mathsf{T}]$$

$$\frac{[\Gamma \succ \Delta, \mathsf{T}A]}{[\Gamma \succ \Delta, \mathsf{T}\mathsf{T}A]}\ [\mathsf{T}\mathsf{T}R] \qquad \frac{[\Gamma, \mathsf{T}A \succ \Delta]}{[\Gamma, \mathsf{T}\mathsf{T}A \succ \Delta]}\ [\mathsf{T}\mathsf{T}L]$$

Negation Rules

$$\frac{[\Gamma, \mathsf{T}A \succ\ ]}{[\Gamma \succ \neg A]}\ [\neg R] \qquad \frac{[\Gamma, A \succ\ ]}{[\Gamma \succ \mathsf{T}\neg A]}\ [\neg R\mathsf{T}] \qquad \frac{[\Gamma \succ \Delta, A]}{[\Gamma, \mathsf{T}\neg A \succ \Delta]}\ [\neg L\mathsf{T}] \qquad \frac{[\Gamma \succ \Delta, \mathsf{T}A]}{[\Gamma, \neg A \succ \Delta]}\ [\neg L]$$

Figure 1: The proof system $\mathcal{SCLP}^{\mathsf{T}}$for the Constructive Logic of Paradox with Strict Truth

2. If $A_1 \wedge A_2 \in r$, then $A_i \in r$ for some $i \in \{1, 2\}$.
3. If $A_1 \vee A_2 \in l$, then $A_i \in l$ for some $i \in \{1, 2\}$.
4. If $A_1 \vee A_2 \in r$, then $A_1 \in r$ and $A_2 \in r$.
5. If $\mathsf{T}(A_1 \wedge A_2) \in l$, then $\mathsf{T}(A_1) \in l$ and $\mathsf{T}(A_2) \in l$.
6. If $\mathsf{T}(A_1 \wedge A_2) \in r$, then $\mathsf{T}(A_i) \in r$ for some $i \in \{1, 2\}$.
7. If $\mathsf{T}(A_1 \vee A_2) \in l$, then $\mathsf{T}(A_i) \in l$ for some $i \in \{1, 2\}$.
8. If $\mathsf{T}(A_1 \vee A_2) \in r$, then $\mathsf{T}(A_1) \in r$ and $\mathsf{T}(A_2) \in r$.
9. If $\neg A \in l$, then $\mathsf{T}(A) \in r$.
10. If $\mathsf{T}(\neg A) \in l$ then $A \in r$.
11. If $\mathsf{T}(A) \in l$, then $A \in l$.
12. If $\mathsf{TT}(A) \in l$, then $\mathsf{T}(A) \in l$.
13. If $\mathsf{TT}(A) \in r$, then $\mathsf{T}(A) \in r$.

Definition 5 (T-complexity)**.** Given a formula A, let the T-complexity $tc(A)$ be defined as follows.

1. $tc(p) = 0$
2. $tc(\mathsf{T}p) = 0$
3. $tc(\neg A) = 1 + tc(A)$
4. $tc(\mathsf{T}\neg A) = 1 + tc(A)$
5. $tc(\mathsf{TT}A) = 1 + tc(A)$
6. $tc(A \wedge B) = tc(\mathsf{T}(A \wedge B)) = 1 + tc(A) + tc(B)$
7. $tc(A \vee B) = tc(\mathsf{T}(A \vee B)) = 1 + tc(A) + tc(B)$

Lemma 6. *Any unprovable sequent $[\Gamma \succ \Delta]$ can be expanded to a T-saturated pair (l, r) s.t. $\Gamma \subseteq l$ and $\Delta \subseteq r$.*

Definition 6 (T-Canonical Model)**.** The T-canonical model $\mathcal{M}^{\mathsf{T}} = \langle W, \leq, V_s, V_t \rangle$ is defined by letting W be the set of all T-saturated pairs (l, r), $(l, r) \leq (l', r')$ iff $l \subseteq l'$, and by letting $(l, r) \in V_t(p_i)$ iff $p_i \in l$ and $(l, r) \in V_s(p_i)$ iff $\mathsf{T}(p_i) \in l$.

It is easy to see that a T-canonical model is indeed a model, the only non-obvious condition being that it satisfies CONTAINMENT, but this follows directly from Definition 4.11.

Theorem 10. *For any* T*-saturated pair* $(l, r) \in W$ *in* $\mathcal{M}^{\mathsf{T}}$ *we have:*

- *If* $A \in l$*, then* $(l, r) \Vdash_t A$.
- *If* $\mathsf{T}(A) \in l$*, then* $(l, r) \Vdash_s A$.
- *If* $A \in r$*, then* $(l, r) \nVdash_t A$.
- *If* $\mathsf{T}(A) \in r$*, then* $(l, r) \nVdash_s A$.

Proof. By induction on the T-complexity of A. For the basis case, it follows from the definition of the T-canonical model that if $p_i \in l$ then $(l, r) \in V_t(p_i)$ and so $(l, r) \Vdash_t p_i$, and likewise for $\mathsf{T}p_i \in l$. Suppose then, that $p_i \in r$. As (l, r) is a T-saturated pair it follows that $p_i \notin l$, as otherwise $[l \succ r]$ would be derivable, and so $(l, r) \notin V_t(p_i)$. The case when $\mathsf{T}p_i \in r$ is similar.

For the induction step, we treat the cases where $A = B \wedge C$, $A = \mathsf{TT}B$, and $A = \neg B$. We begin with the cases where $A = B \wedge C$.

$B \wedge C \in l$: If $B \wedge C \in l$, then by Definition 4.1 we have $B \in l$ and $C \in l$. So by the induction hypothesis, we have $(l, r) \Vdash_t B$ and $(l, r) \Vdash_t C$ and so $(l, r) \Vdash_t B \wedge C$.

$\mathsf{T}(B \wedge C) \in l$: If $\mathsf{T}(B \wedge C) \in l$, then by Definition 4.5, we have $\mathsf{T}B \in l$ and $\mathsf{T}C \in l$. So by the induction hypothesis, we have $(l, r) \Vdash_s B$ and $(l, r) \Vdash_s C$ and so $(l, r) \Vdash_s B \wedge C$.

$B \wedge C \in r$: If $B \wedge C \in r$, then by Definition 4.2, we must have either $B \in r$ or $C \in r$. So by the induction hypothesis, we have $(l, r) \nVdash_t B$ or $(l, r) \nVdash_t C$, and so either way we have $(l, r) \nVdash_t B \wedge C$.

$\mathsf{T}(B \wedge C) \in r$: If $\mathsf{T}(B \wedge C) \in r$, then by Definition 4.6, we must have either $\mathsf{T}B \in r$ or $\mathsf{T}C \in r$. So by the induction hypothesis, we have either $(l, r) \nVdash_s B$ or $(l, r) \nVdash_s C$, and so either way we have $(l, r) \nVdash_s B \wedge C$.

Consider now, the case where $A = \mathsf{TT}B$.

$\mathsf{TT}B \in l$: If $\mathsf{TT}B \in l$ then by Definition 4.12, we have $\mathsf{T}B \in l$ and so by the induction hypothesis, we have $(l, r) \Vdash_s B$ and so by our truth conditions, we have $(l, r) \Vdash_s \mathsf{T}(B)$ and so $(l, r) \Vdash_t \mathsf{TT}(B)$ as desired.

$\mathsf{TT}B \in r$**:** If $\mathsf{TT}B \in r$ then by Definition 4.13, we have $\mathsf{T}B \in r$ and so by the induction hypothesis, we have $(l, r) \nVdash_s B$ and so by our truth conditions we have $(l, r) \nVdash_s \mathsf{T}(B)$ and so $(l, r) \nVdash_t \mathsf{TT}(B)$ as desired.

$\mathsf{TTT}B \in l$**:** If $\mathsf{TTT}B \in l$ then by Definition 4.12, we have $\mathsf{TT}B \in l$ and so by Definition 4.12 again we have $\mathsf{T}B \in l$. By the induction hypothesis, it follows that $(l, r) \Vdash_s B$ and so $(l, r) \Vdash_s \mathsf{TT}B$ by our truth conditions for T.

$\mathsf{TTT}B \in r$**:** If $\mathsf{TTT}B \in r$ then by Definition 4.13, we have $\mathsf{TT}B \in r$ and so by Definition 4.13 again we have $\mathsf{T}B \in r$. By the induction hypothesis, it follows that $(l, r) \nVdash_s B$ and so $(l, r) \nVdash_s \mathsf{TT}B$, by our truth conditions for T.

Consider now, the case where $A = \neg B$.

$\neg B \in l$**:** If $\neg B \in l$ then for every accessible (l', r'), we must also have $\neg B \in l$ and so by Definition 4.9 $\mathsf{T}(B) \in r$. So by the induction hypothesis, for all such (l', r') we have $(l', r') \nVdash_s B$, and so $(l, r) \Vdash_t \neg B$.

$\mathsf{T}\neg B \in l$**:** If $\mathsf{T}\neg B \in l$ then we have $\mathsf{T}\neg B \in l'$ for all accessible (l', r'). So by Definition 4.10 we have $B \in r'$ and so by the induction hypothesis, $(l', r') \nVdash_t B$, and so $(l, r) \Vdash_s \neg B$.

$\neg B \in r$**:** We show that $[l, \mathsf{T}B \succ\ ]$ is unprovable. If not then by $[\neg R]$ and $[KR]$ we would have $[l \succ r]$ provable, contradicting T-saturation. So we extend $(l \cup \{\mathsf{T}(B)\}, \emptyset)$ to a T-saturated pair (l', r') using Lemma 6. Then we have $(l, r) \leq (l', r')$. As $\mathsf{T}(B) \in l'$ we have $(l', r') \Vdash_s B$, an so $(l, r) \nVdash_t \neg B$.

$\mathsf{T}\neg B \in r$**:** We show $[l, B \succ\ ]$ is unprovable. If not then by $[\neg R\mathsf{T}]$ we would have $[l \succ \mathsf{T}\neg B]$ provable, and so by weakening $[l \succ r]$ provable, contradicting T-saturation. So we extend $(l \cup \{B\}, \emptyset)$ to a T-saturated pair (l', r') using Lemma 6. Then we have $(l, r) \leq (l', r')$, and as $B \in l$ by the induction hypothesis, $(l', r') \Vdash_t B$, and so $(l, r) \nVdash_s \neg B$. □

Given this we can prove completeness.

Theorem 11 (Completeness for $\mathcal{L}_\mathsf{T}$)**.** *If the $\mathcal{L}_\mathsf{T}$-sequent $[\Gamma \succ \Delta]$ is valid in the class of all models, then $[\Gamma \succ \Delta]$ is derivable in $\mathcal{SCLP}^\mathsf{T}$.*

Proof. We prove the contrapositive. Suppose that $[\Gamma \succ \Delta]$ is not derivable in $\mathcal{SCLP}^\mathsf{T}$. Then by lemma 6, there is a T-saturated pair (l, r) where $\Gamma \subseteq l$ and $\Delta \subseteq r$. So by Theorem 10, we have that in the T-canonical model $\mathcal{M}^\mathsf{T}$ that $\mathcal{M}^\mathsf{T}, (l, r) \Vdash_t C$ for all $C \in \Gamma$, and that $\mathcal{M}^\mathsf{T}, (l, r) \nVdash_t D$ for all $D \in \Delta$, and so the sequent is invalid. □

Interestingly, we have bought completeness here at the cost of (strong) paraconsistency. The essential problem here is that using T we can define an intuitionistic negation as $\mathsf{T}\neg A$. As explosion is valid for intuitionistic negation our logic fails to be strongly paraconsistent.

Lemma 7. $\vdash_{\mathsf{cLP}^{\mathsf{T}}}$ *is not paraconsistent.*

Proof. To see this note that the sequent $[A, \mathsf{T}\neg A \succ\ ]$ is derivable, as shown below:

$$\dfrac{\dfrac{}{[A \succ A]}\ [Id]}{[A, \mathsf{T}\neg A \succ\ ]}\ [\neg LT]$$

So by Theorem 11, we have $A, \mathsf{T}\neg A \vdash_{\mathsf{cLP}^{\mathsf{T}}} \emptyset$. □

It is easy to show that the disjunction property still holds for $\vdash_{\mathsf{cLP}^{\mathsf{T}}}$, though, using the same proof as was given in §2. So as we can see here, we can buy completeness for cLP, but at the cost of it no longer being paraconsistent in the extended language.

4 cLP with a De Morgan Negation: Paraconsistent, but not Constructive

In this section, we will consider the addition of a De Morgan negation '$\sim$' to cLP with the following clauses:

- $\mathcal{M}, w \Vdash_s {\sim}A$ iff $\mathcal{M}, w \nVdash_t A$.
- $\mathcal{M}, w \Vdash_t {\sim}A$ iff $\mathcal{M}, w \nVdash_s A$.

As before we will let $\mathcal{L}^{\sim}$ be the language which results from the addition of $\sim$ to $\mathcal{L}$, and say that a sequent $[\Gamma \succ \Delta]$ from the language $\mathcal{L}^{\sim}$ is valid in $\mathsf{cLP}^{\sim}$ iff it is valid on all frames, sometimes writing this as $\Gamma \vdash_{\mathsf{cLP}^{\sim}} \Delta$. The proof system $\mathcal{SCLP}^{\sim}$for this logic consists of the rules and basic sequents in Figure 2.

One curious feature of this extension is that it is no longer the case that all formulas are persistent. In particular, formulas which contain the De Morgan negation are not persistent. This is reflected in the $[{\sim}\neg L]$ and $[\neg R]$ rules of $\mathcal{SCLP}^{\sim}$. Consider, for example the rule $[\neg R]$, where $\overrightarrow{A_i}$ is a metavariable over sequences of formulas $A_1, \ldots, A_n$ for some n.

$$\dfrac{[\overrightarrow{\neg A_i}, \overrightarrow{p_i} \succ \overrightarrow{{\sim}\neg A_i}, \overrightarrow{{\sim} p_i}, {\sim}A]}{[\overrightarrow{\neg A_i}, \overrightarrow{p_i} \succ \overrightarrow{{\sim}\neg A_i}, \overrightarrow{{\sim} p_i}, \neg A]}\ [\neg R]$$

Initial Sequents

$$\frac{}{[\Gamma, A \succ A, \Delta]}\ [Id] \qquad \frac{}{[\Gamma \succ \Delta, A, {\sim}A]}\ [{\sim}LEM]$$

Structural Rules

$$\frac{[\Gamma \succ \Delta, A] \quad [A, \Gamma' \succ \Delta']}{[\Gamma, \Gamma' \succ \Delta, \Delta']}\ [Cut]$$

$$\frac{[\Gamma \succ \Delta]}{[A, \Gamma \succ \Delta]}\ [KL] \qquad \frac{[\Gamma \succ \Delta]}{[\Gamma \succ \Delta, A]}\ [KR] \qquad \frac{[A, A, \Gamma \succ \Delta]}{[A, \Gamma \succ \Delta]}\ [WL] \qquad \frac{[\Gamma \succ \Delta, A, A]}{[\Gamma \succ \Delta, A]}\ [WR]$$

Operational Rules

$$\frac{[\Gamma, A, B \succ \Delta]}{[\Gamma, A \wedge B \succ \Delta]}\ [\wedge L] \qquad \frac{[\Gamma \succ \Delta, A] \quad [\Gamma' \succ \Delta', B]}{[\Gamma, \Gamma' \succ \Delta, \Delta', A \wedge B]}\ [\wedge R]$$

$$\frac{[\Gamma, A \succ \Delta] \quad [\Gamma', B \succ \Delta']}{[\Gamma, \Gamma', A \vee B \succ \Delta, \Delta']}\ [\vee L] \qquad \frac{[\Gamma \succ \Delta, A, B]}{[\Gamma \succ \Delta, A \vee B]}\ [\vee R]$$

De Morgan Rules

$$\frac{[\Gamma \succ \Delta, {\sim}A] \quad [\Gamma \succ \Delta, {\sim}B]}{[\Gamma \succ \Delta, {\sim}(A \vee B)]}\ [{\sim} \vee R] \qquad \frac{[\Gamma, {\sim}A, {\sim}B \succ \Delta]}{[\Gamma, {\sim}(A \vee B) \succ \Delta]}\ [{\sim} \vee L]$$

$$\frac{[\Gamma, {\sim}A \succ \Delta] \quad [\Gamma, {\sim}B \succ \Delta]}{[\Gamma, {\sim}(A \wedge B) \succ \Delta]}\ [{\sim} \wedge L] \qquad \frac{[\Gamma \succ \Delta, {\sim}A, {\sim}B]}{[\Gamma \succ \Delta, {\sim}(A \wedge B)]}\ [{\sim} \wedge R]$$

$$\frac{[\Gamma \succ A, \Delta]}{[\Gamma, {\sim}{\sim}A \succ \Delta]}\ [{\sim}{\sim}L] \qquad \frac{[\Gamma \succ \Delta, A]}{[\Gamma \succ \Delta, {\sim}{\sim}A]}\ [{\sim}{\sim}R]$$

Negation Rules

$$\frac{[A, \overrightarrow{\neg A_i}, \overrightarrow{p_i} \succ \overrightarrow{{\sim}\neg A_i}, \overrightarrow{{\sim}p_i}]}{[{\sim}\neg A, \overrightarrow{\neg A_i}, \overrightarrow{p_i} \succ \overrightarrow{{\sim}\neg A_i}, \overrightarrow{{\sim}p_i}]}\ [{\sim}\neg L] \qquad \frac{[\overrightarrow{\neg A_i}, \overrightarrow{p_i} \succ \overrightarrow{{\sim}\neg A_i}, \overrightarrow{{\sim}p_i}, {\sim}A]}{[\overrightarrow{\neg A_i}, \overrightarrow{p_i} \succ \overrightarrow{{\sim}\neg A_i}, \overrightarrow{{\sim}p_i}, \neg A]}\ [\neg R]$$

$$\frac{[\Gamma \succ A, \Delta]}{[\Gamma \succ {\sim}\neg A, \Delta]}\ [{\sim}\neg R] \qquad \frac{[\Gamma, {\sim}A \succ \Delta]}{[\Gamma, \neg A \succ \Delta]}\ [\neg L]$$

Figure 2: The Proof System $\mathcal{SCLP}^{\sim}$for The Constructive Logic of Paradox with De Morgan Negation.

The non-principal formulas of this rule which occur on the left hand side of the sequent (i.e., formulas of the form $\neg A$ or p_i) are all formulas which are T-PERSISTENT, while those on the right hand side (i.e., formulas of the form $\sim\neg A$ and $\sim p_i$) are T-ANTI-PERSISTENT, where A is X-ANTI-PERSISTENT iff if $w \nVdash_x A$ and $w \leq v$, then $v \nVdash_x A$.

As before soundness is a relatively routine matter.

Theorem 12 (Soundness for $\mathcal{SCLP}^{\sim}$)**.** *If a sequent* $[\Gamma \succ \Delta]$ *is derivable in* $\mathcal{SCLP}^{\sim}$ *then it is valid in* $\mathsf{cLP}^{\sim}$*.*

Proof. By induction on the length of derivations. For the basis case it is easy to see that $[Id]$ is valid. For the case of $[LEM\sim]$, suppose that for some model $\mathcal{M} = \langle W, \leq, V_s, V_t\rangle$ and $w \in W$ we have $\mathcal{M}, w \Vdash_t C$ for all $C \in \Gamma$ and $\mathcal{M}, w \nVdash_t \sim A$. Then we have $\mathcal{M}, w \Vdash_s A$ and so by Lemma 2, $\mathcal{M}, w \Vdash_t A$, as desired.

In the induction step, we treat $[\neg R]$, the other cases either being routine or similar. We show that if the conclusion sequent fails to hold in a model, then so does the premise sequent. Suppose, then, that we have a model $\mathcal{M} = \langle W, \leq, V_s, V_t\rangle$ and a $w \in W$ such that (i) $\mathcal{M}, w \Vdash_t \overrightarrow{\neg A_i}$, (ii) $\mathcal{M}, w \Vdash_t \overrightarrow{p_i}$, (iii) $\mathcal{M}, w \nVdash_t \overrightarrow{\sim\neg A_i}$, (iv) $\mathcal{M}, w \nVdash_t \overrightarrow{\sim p_i}$, and (v) $\mathcal{M}, w \nVdash_t \neg A$. From (v) it follows that for some $v \geq w$ we have $\mathcal{M}, v \Vdash_s A$, from which it follows that $\mathcal{M}, v \nVdash_t \sim A$. But as the formulas in (i) and (ii) are T-PERSISTENT they also hold at v, and as the formulas in (iii) and (iv) are T-ANTI-PERSISTENT they also fail at v and so the premise sequent fails to hold at v, and so fails to hold in $\mathcal{M}$. $\square$

To prove completeness in this setting we will draw on some techniques (particularly concerning how to treat rules containing non-persistent formulas) used in [12] to prove completeness for the logic **C+J**, which combines classical and intuitionistic logic.

Definition 7. Say that a pair of sets of formulas (l, r) is *d-saturated* iff $[l \succ r]$ is unprovable and we have the following:

1. If $A_1 \wedge A_2 \in l$ then $A_1 \in l$ and $A_2 \in l$.
2. If $A_1 \wedge A_2 \in r$ then $A_i \in r$ for some $i \in \{1, 2\}$.
3. If $\sim(A_1 \wedge A_2) \in l$ then $\sim A_i \in l$ for some $i \in \{1, 2\}$.
4. If $\sim(A_1 \wedge A_2) \in r$ then $\sim A_1 \in r$ and $\sim A_2 \in r$.
5. If $A_1 \vee A_2 \in l$ then $A_i \in l$ for some $i \in \{1, 2\}$.
6. If $A_1 \vee A_2 \in r$ then $A_1 \in r$ and $A_2 \in l$.

7. If $\sim(A_1 \vee A_2) \in l$ then $\sim A_1 \in l$ and $\sim A_2 \in l$.
8. If $\sim(A_1 \vee A_2) \in r$ then $\sim A_i \in r$ for some $i \in \{1, 2\}$.
9. If $\sim\sim A \in l$ then $A \in l$.
10. If $\sim\sim A \in r$ then $A \in r$.
11. If $\sim A \in r$, then $A \in l$.
12. If $A \in r$, then $\sim A \in l$.
13. If $\neg A \in l$ then $\sim A \in l$.
14. If $\sim\neg A \in r$ then $A \in r$.

Lemma 8. *Any unprovable sequent* $[\Gamma \succ \Delta]$ *can be expanded to a d-saturated pair* (l, r) *s.t.* $\Gamma \subseteq l$ *and* $\Delta \subseteq r$.

Definition 8. The d-canonical model $\mathcal{M}^d = \langle W, \leq, V_t, V_s \rangle$ is defined by letting W be the set of all saturated pairs (l, r) of formulas, and letting $(l, r) \leq (l', r')$ whenever we have all of the following conditions met:

1. If $p \in l$ then $p \in l'$.
2. If $\sim p \in r$ then $\sim p \in r'$.
3. If $\neg A \in l$ then $\neg A \in l'$.
4. If $\sim\neg A \in r$ then $\sim\neg A \in r'$.

Finally, let $(l, r) \in V_t(p_i)$ iff $p_i \in l$ and $(l, r) \in V_s(p_i)$ iff $\sim p_i \in r$.

Again to see that the d-canonical model is a model the only non-obvious condition which we need to check is that it satisfies CONTAINMENT. In this case suppose $(l, r) \in V_s(p_i)$, and so $\sim p_i \in r$. Then by Definition 7.11, it follows that $p_i \in l$ and, and so $(l, r) \in V_t(p_i)$, as desired.

Definition 9. Given a formula A, let the *De Morgan Complexity* $dmc(A)$ be defined as follows.

1. $dmc(p) = 0$
2. $dmc(\sim p) = 0$
3. $dmc(\neg A) = 1 + dmc(A)$

4. $dmc(\sim\neg A) = 1 + dmc(A)$

5. $dmc(\sim\sim A) = 1 + dmc(A)$

6. $dmc(A \wedge B) = dmc(\sim(A \wedge B)) = 1 + dmc(A) + dmc(B)$

7. $dmc(A \vee B) = dmc(\sim(A \vee B)) = 1 + dmc(A) + dmc(B)$

Note that it follows from our definition of De Morgan complexity that $dmc(\sim A)$ is $dmc(\neg A) - 1$. We will appeal to this fact in a few places in the following theorem.

Theorem 13. *For any d-saturated pair $(l, r) \in W$ in $\mathcal{M}^d$ we have:*

1. *If $A \in l$ then $(l, r) \Vdash_t A$.*

2. *If $\sim A \in l$ then $(l, r) \nVdash_s A$.*

3. *If $A \in r$ then $(l, r) \nVdash_t A$.*

4. *If $\sim A \in r$ then $(l, r) \Vdash_s A$.*

Proof. By induction on $dmc(A)$. For the basis case, cases 1 and 4 are covered by the definition of V_s and V_t. For case 2, if $\sim p_i \in l$ then as (l, r) is d-saturated it follows that $\sim p_i \notin r$, and so $(l, r) \notin V_s(p_i)$, and thus $(l, r) \nVdash_s p_i$. 3 follows by similar reasoning.

The only remaining cases of interest are those for negation, and those involving De Morgan negation. Here we will treat those for negation and for the De Morgan negation of negations. We begin with **negation**.

1. Suppose $\neg A \in l$. Consider any point (l', r') such that $(l, r) \leq (l', r')$. Then by definition $\neg A \in l'$. So by Definition 7.13, we have that $\sim A \in l'$. As $dmc(\sim A) = dmc(\neg A) - 1$, we can appeal to induction hypothesis 2, allowing us to conclude that $(l', r') \nVdash_s A$. But (l', r') was arbitrary, and so we have $(l, r) \Vdash_t \neg A$.

2. Suppose that $\sim\neg A \in l$. We show that $[\Gamma' \succ \Delta'] =$

$$[\{p_i \colon p_i \in l\} \cup \{\neg A_i \colon \neg A_i \in l\} \cup \{A\} \succ \{\sim\neg A_i \colon \sim\neg A_i \in r\} \cup \{\sim p_i \colon \sim p_i \in r\}]$$

is unprovable. If not then by $[\sim\neg R]$ and $[K]$ we would have (l, r) provable contradicting saturation. So extend $[\Gamma' \succ \Delta']$ to a saturated set $(l', r') \in W$. By definition $(l, r) \leq (l', r')$. As $A \in l$ by 1 and the hypothesis of induction, we have $(l', r') \Vdash_t A$. So we have $(l, r) \nVdash_s \neg A$ as desired.

3. Suppose that $\neg A \in r$. We show that $[\Gamma' \succ \Delta'] =$

   $$[\{p_i \colon p_i \in l\} \cup \{\neg A_i \colon \neg A_i \in l\} \succ \{{\sim}\neg A_i \colon {\sim}\neg A_i \in r\} \cup \{{\sim}p_i \colon {\sim}p_i \in r\} \cup \{{\sim}A\}]$$

   is unprovable. If not then $[\Gamma' \succ \Delta']$ is provable, and so by $[\neg R]$ and $[K]$ we have $[l \succ r]$ provable, contradicting saturation! So $[\Gamma' \succ \Delta']$ is unprovable and so we can extend it to a saturated set $(l', r') \in W$. By construction, $(l, r) \leq (l', r')$ and ${\sim}A \in r'$ and so, as $dmc({\sim}A) = dmc(\neg A) - 1$, by 4 and the inductive hypothesis, we have $(l', r') \Vdash_s A$, and so $(l, r) \nVdash_t \neg A$.

4. Suppose ${\sim}\neg A \in r$. Then consider a point (l', r') where $(l, r) \leq (l', r')$. By definition we have ${\sim}\neg A \in r'$ and so by Definition 7.14 $A \in r'$. So by 3 and the inductive hypothesis, we have $(l', r') \nVdash_t A$. So it follows that $(l, r) \Vdash_s \neg A$.

Now for the **De Morgan negation of negation**

1. Suppose ${\sim}\neg A \in l$. Then we can proceed as in case 2 for negation to conclude that $(l, r) \nVdash_s \neg A$, and thus $(l, r) \Vdash_t {\sim}\neg A$, as desired.

2. Suppose ${\sim}{\sim}\neg A \in l$. Then by Definition 7.9, we have $\neg A \in l$, and so by case 1 for negation above $(l, r) \Vdash_t \neg A$ and so $(l, r) \nVdash_s {\sim}\neg A$, as desired.

3. Suppose that ${\sim}\neg A \in r$. Then by case 4 for negation, we have $(l, r) \Vdash_s \neg A$, and so $(l, r) \nVdash_t {\sim}\neg A$, as desired.

4. Suppose that ${\sim}{\sim}\neg A \in r$. Then by Definition 7.10, we have $\neg A \in r$. So by case 3 for negation above we have $(l, r) \nVdash_t \neg A$, and so $(l, r) \Vdash_s {\sim}\neg A$, as desired. □

Theorem 14 (Completeness for $\mathcal{SCLP}^{\sim}$)**.** *If a sequent is valid in* $\mathsf{cLP}^{\sim}$*, then it is derivable in* $\mathcal{SCLP}^{\sim}$*.*

Again, we have bought our completeness at a cost, in this case the cost of constructivity.

Lemma 9. $\vdash_{\mathsf{cLP}^{\sim}}$ *is not constructive.*

Proof. In this case, we do not have the disjunction property, as we have $\vdash_{\mathsf{cLP}^{\sim}} p \vee {\sim}p$ but neither $\vdash_{\mathsf{cLP}^{\sim}} p$ nor $\vdash_{\mathsf{cLP}^{\sim}} {\sim}p$. □

It is easy to show though that $\vdash_{\mathsf{cLP}^{\sim}}$ is still paraconsistent, again using essentially the same proof as we used in §2 but conflating the distinction between the negation of cLP and our De Morgan negation. So we have bought completeness for cLP , but this time at the cost of it not longer being constructive in the extended language.

5 Conclusion

In this paper, we introduced a logic cLP, the constructive logic of paradox, and showed that it bore strong affinities to both intuitionistic logic and the logic of paradox. We have been unable to prove completeness for this logic in the standard signature, although we were able to prove completeness for two different extensions of our base language. In the first case, we showed that we could prove completeness if we enriched our language by the addition of a 'strict truth' operator, moving to a signature more similar to those used by Woodruff in [13, 14], although at the cost of paraconsistency of the logic in the extended language. In the second case, we showed that we could prove completeness if we enriched the language by the addition of a De Morgan negation, although this time at the cost of the constructivity of the logic in the extended language.

What should we make of this seeming incompleteness of cLP in the base language? Incompleteness has its source in the semantics for a logic being too expressive for the deductive apparatus of a logic. One result which is very suggestive that this is what is going on in the present setting is the following.

Proposition 2. *The logics* $\mathsf{cLP}^{\mathsf{T}}$ *and* $\mathsf{cLP}^{\sim}$ *are both conservative extensions of* cLP*.*

Proof. This result follows from the fact that models for $\mathsf{cLP}^{\mathsf{T}}$ and $\mathsf{cLP}^{\sim}$ are just models for cLP with the addition of new truth-in-a-model clauses for the new operators. $\square$

Our inability to prove completeness in the standard signature, combined with the fact that we are able to prove it in a semantically-conservative extension of our language, suggests that our semantics is too expressive for us to be able to treat it in the standard signature. This thought is reinforced by the fact that the semantics for non-transitive constructive logics seem to need to resort to either partial semantic clauses (as is done in [11]), or labelled proof-systems (as is done in [7]). We leave further consideration of this issue for future work.

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