

# Invitation to Constructive Nonreflexive and Nontransitive Logics

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**Abstract.** Rejecting structural principles, like the reflexivity or transitivity of the entailment relation, provides an attractive solution to the semantic paradoxes, allowing us to give a uniform analysis and solution to the paradoxes of self-reference without having to engage in piecemeal rejection of inferential principles governing the sentential connectives. To date all of the proposed nonreflexive and nontransitive logics have been substructural companions of classical logic. In this paper we correct this by investigating the substructural companions of intuitionistic logic, and their relation to their classical counterparts.

**Keywords:** Substructural Logics · Metainferences · Intuitionistic Logic.

## 1 Motivation and Preliminaries

According to substructural approaches to the paradoxes of self-reference, rather than addressing paradoxes like the Liar or Curry’s Paradox by rejecting inferences involving negation or the conditional, we should instead “grapple with the paradoxes where they live: in the basic features of argumentation” [16, p.310], by rejecting *structural rules*—schematic rules of inference which encode basic structural features of arguments themselves. Our focus in the present paper will be nonreflexive and nontransitive substructuralist theories, which advise us to reject the claim that all sentences entail themselves (as encoded in the principle  $[Id]$ ), or that arguments can be chained together (as encoded in the principle  $[Cut]$ ).<sup>1,2</sup>

$$\frac{}{[A \succ A]} [Id] \quad \frac{[\Gamma \succ \Delta, A] \quad [A, \Gamma' \succ \Delta']}{[\Gamma, \Gamma' \succ \Delta, \Delta']} [Cut]$$

The most prominent nonreflexive and nontransitive propositional logics are the logics dubbed TS and ST in [3]. Throughout we assume that our language  $\mathcal{L}$

<sup>1</sup> Nonreflexive approaches to paradox have been argued for in [7] and [6], and nontransitive approaches in [3], [14], [15].

<sup>2</sup> We will deal solely with propositional languages. Given a propositional language  $\mathcal{L}$  a *sequent* is a pair  $\langle \Gamma, \Delta \rangle$  of finite sets of formulas taken from  $\mathcal{L}$ , which we will typically write as  $[\Gamma \succ \Delta]$ . We will also be concerned throughout with metainferences, with metasequents standing to metainferences in the same way sequents stand to inferences, with a *metasequent* being defined as an ordered pair  $\langle \Sigma, \sigma \rangle$  of a finite set of sequents  $\Sigma$  and a sequent  $\sigma$ , which we write as  $\Sigma \Rightarrow \sigma$ .

is an absolutely free-algebra of formulas with a set  $\text{Atom}$  of countably many generators  $p_1, p_2, p_3 \dots$  (the first three of which we abbreviate  $p, q, r$ ) and the primitive singularly connective  $\neg$ , and primitive binary connectives  $\wedge, \vee, \rightarrow$ . Following [3] we can characterize these logics semantically in terms of sets of trivaluations, functions from our propositional language  $\mathcal{L}$  to the space of values  $\{1, i, 0\}$  which interpret the sentential connectives as indicated by the matrices in Figure 1.<sup>3</sup>

	$\neg$	$\wedge$	1	$i$	0	$\vee$	1	$i$	0	$\rightarrow$	1	$i$	0
1	0	1	1	$i$	0	1	1	1	1	1	1	$i$	0
$i$	$i$	$i$	$i$	$i$	0	$i$	1	$i$	$i$	$i$	1	$i$	$i$
0	1	0	0	0	0	0	1	$i$	0	0	1	1	1

**Fig. 1.** The Strong Kleene Matrices

Say that a trivaluation is a Strong Kleene valuation whenever it agrees with the matrices in Figure 1. Then we can then characterize the sequents which are valid according to both TS and ST as follows.

**Definition 1.** Let  $T = \{1\}$  and  $S = \{1, i\}$ . Then a Strong Kleene valuation  $v$  is an  $XY$ -counterexample to the sequent  $[Γ \succ Δ]$  iff  $v[Γ] \subseteq X$  and  $v[Δ] \cap Y = \emptyset$ , where  $X, Y \in \{S, T\}$ . A sequent  $[Γ \succ Δ]$  which has no  $XY$ -counterexamples is  $XY$ -valid.

As noted in [3], the notions of  $SS$  and  $TT$  validity determine Strong Kleene logic and the Logic of Paradox. The more interesting notions are the ‘mixed’ ones. As sets of arguments the  $TS$ - and  $ST$ -valid arguments are not that distinctive—the  $ST$ -valid arguments are all and only the classically valid arguments, and set of  $TS$ -valid arguments is empty. As is pointed out in [7] in the context of TS, though, the best way to understand these logics is in terms of their valid *metainferences*—inferences which have inferences as both premises and conclusions. Say that a Strong Kleene valuation  $v$  is an  $XY$ -counterexample to a metainference  $\Sigma \Rightarrow \sigma$  iff it is not an  $XY$ -counterexample to any sequent in  $\Sigma$ , but is an  $XY$ -counterexample to  $\sigma$ . Then we can say that a metainference is  $XY$ -(locally) valid iff there it has no  $XY$ -counterexamples. It is routine to see that there are  $ST$ -counterexamples to the metainference corresponding to  $[Cut]$ , making the set of  $ST$ -valid metainferences distinct from those for classical logic, while there is no  $TS$ -counterexample to the metainference  $[p \succ p], [q \succ q] \Rightarrow [p, q \succ p \wedge q]$ , distinguishing the  $TS$ -valid metainferences from the empty set.

Of particular interest to us here is the fact that TS and ST are, in a very precisely specifiable way, nonreflexive and nontransitive versions of classical logic—in fact [15] has even argued that in the present signature ST is nothing more

<sup>3</sup> For a thorough investigation of the valuational model theory of nonreflexive and nontransitive logics the interested reader should consult [10].

than classical logic. To make this more precise, let us temporarily identify a logic  $L$  with a collection of valuations  $V_L$  paired with a counterexample relation  $\ast_L$  on sequents. Then, when  $R$  is a metasequent, we can say that a logic  $L_1$  is the  $R$ -companion of a logic  $L_2$  whenever (i)  $R$  is not locally valid according to  $\ast_{L_2}$ , and (ii) the set of metasequents which are valid according to  $\ast_{L_2}$  when restricting to valuations in  $V$  which are not  $\ast_{L_2}$ -counterexamples to  $R$  is identical to those metasequents which are valid according to  $\ast_{L_1}$ .

**Theorem 1.** *Classical logic is the  $[Id]$ -companion of  $TS$ , and the  $[Cut]$ -companion of  $ST$ .*

*Proof.* We prove the first of these. As noted above,  $[Id]$  is not  $TS$ -valid. To show that the restriction to Strong Kleene valuations which  $TS$ -validate  $[Id]$  determines classical logic, consider that for any trivaluation  $v$ , whenever  $v(A) = i$  it follows that  $v$  is a  $TS$ -counterexample to  $[A \succ A]$ . So the trivaluations which  $TS$ -validate all instances of  $[Id]$  are those valuations which never assign a formula  $i$ , which are precisely the Boolean valuations.  $\square$

This result points to one of the key advantages of substructural approaches to the paradoxes. Substructural solutions let us very clearly point out both what the mistake involved in paradoxical reasoning is, as well as what the correct canons of reasoning are when that mistake is not relevant in a relatively modular way. For example, according to the nonreflexive logician the mistake in paradoxical reasoning is thinking that paradoxical sentences entail themselves. Given the above result, the nonreflexive logician who thinks that, outside of the context of paradox, classical logic gives the canons of correct reasoning, should accept  $TS$  as the correct logic.

What about the nonreflexive and nontransitive logicians who don't wish to hew as closely to classical logic outside of paradoxical contexts, though? The purpose of the present paper is to act as the first steps in the investigation of this question in the context of intuitionistic logic, by investigating the nonreflexive and nontransitive counterparts of intuitionistic logic. The structure of the remainder of the paper is as follows. In section §2 we begin by outlining a modification of the Kripke semantics for intuitionistic logic over which we are able to define analogues of the nonreflexive and nontransitive logics  $TS$  and  $ST$ . In §3 I then outline a labelled metasequent calculus in the sense of [8] which in §4 we prove to be sound & complete for the models defined in §2. Finally in §5 we look at the relations between the nonreflexive and nontransitive companions of classical logic and intuitionistic logic, before concluding in §6.

## 2 Semantics

In the present section we will define a general semantic framework for reasoning about nonreflexive and nontransitive intuitionistic logic. Our present framework is essentially the Kripke semantic version of the topological semantics which

Girard used in [11] in his discussion of the semantics of Cut-elimination in intuitionistic logic.<sup>4</sup> A model for substructural intuitionistic logic (henceforth simply ‘a model’) is a structure  $\langle W, \leq, V_s, V_t \rangle$  where  $W$  is a non-empty set,  $\leq$  a partial order on  $W$ , and  $V_s$  and  $V_t$  are functions from  $\text{Atom}$  to subsets of  $W$  which satisfy the following three conditions:

- CONTAINMENT:  $V_s(p_i) \subseteq V_t(p_i)$
- T-PERSISTENCE: If  $w \in V_t(p_i)$  and  $w \leq v$  then  $v \in V_t(p_i)$ .
- S-PERSISTENCE: If  $w \in V_s(p_i)$  and  $w \leq v$  then  $v \in V_s(p_i)$ .

It can help to have some way of interpreting what’s going on in this semantics. Following the interpretation given in [13], we will think of the members of  $W$  as representing information states, and  $\leq$  as representing the relation of one information state extending another (with possibly further information). The main departure here is to think of information states as providing two different kinds of warrant for statements. The first kind of warrant they can provide is *strict* (or strong) warrant, which we should think of as the kind of warrant for a statement which supports us accepting it. The second kind of warrant a state can provide is *tolerant* (or tentative) warrant, which is sufficient warrant to not reject a statement, but not necessarily sufficient warrant to accept it. On this way of interpreting our semantics CONTAINMENT represents the reasonable thought that a warrant to accept a statement is also a warrant against rejecting it, while S-PERSISTENCE and T-PERSISTENCE record the idea that warrants are preserved under the gaining of new information—one consequence of which is that the tentativeness of tolerant warrant should be thought of as simply a matter of their strength, not whether they can later be overruled.

The distinction between strict and tolerant warrant is reminiscent of the distinction drawn in Artemov and Protopoescu’s Intuitionistic Epistemic Logic [2] between the intuitionistic conceptions of truth and knowledge. Following the BHK-interpretation of the intuitionistic connectives Artemov and Protopoescu identify intuitionistic truth with proof, and characterize intuitionistic knowledge as verification that a proof exists, where proofs are understood to be a species of verification. This results in a novel epistemic logic  $\text{IEL}^-$  which validates the converse of the **T** axiom ( $A \rightarrow \mathbf{K}A$ , where  $\mathbf{K}$  is the knowledge operator), while invalidating the **T** axiom itself (in the form  $\mathbf{K}A \rightarrow A$ ) which is usually taken as characterizing the factivity for knowledge. This failure of intuitionistic knowledge to be factive in this sense is due to the fact that, on the intuitionistic conception advanced by Artemov and Protopoescu, having a verification that a proof exists (as is required for  $\mathbf{K}A$  to hold) doesn’t necessarily mean that one has a proof

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<sup>4</sup> Another similar semantic treatment of nontransitive intuitionistic logic is given in [20]. The main departure from the present approach is that their clause for the evaluation of the conditional is *partial*, only including the ‘only if’ direction of our clause for a conditional *s*-satisfied relative to an information state. We will leave further consideration of the connections between their approach and the present approach for future work.

in hand (as is required for  $A$  to hold).<sup>5</sup> This is similar to our situation, where any statement which has a strict warrant also has a tolerant warrant, but not all statements which have tolerant warrants necessarily have strict warrants. The logics we get by understanding strict warrant as intuitionistic truth, and tolerant warrant as intuitionistic knowledge are significantly weaker than the logics  $\mathbb{L}t$ s and  $\mathbb{L}st$  we characterize below. We leave further investigation of this parallel to future work.

The recursive clauses for evaluating formulas in our language over this models is relativized to the two different sorts of warrants states can provide, with  $x$ -satisfaction of a formula  $A$  at a point  $w$  in a model  $\mathcal{M}$  ( $=\mathcal{M}, w \Vdash_x A$ ), where  $x \in \{s, t\}$ , being defined inductively as follows:

- $\mathcal{M}, w \Vdash_t p_i$  iff  $w \in V_t(p_i)$ .
- $\mathcal{M}, w \Vdash_s p_i$  iff  $w \in V_s(p_i)$ .
- $\mathcal{M}, w \Vdash_t A \wedge B$  iff  $\mathcal{M}, w \Vdash_t A$  and  $\mathcal{M}, w \Vdash_t B$
- $\mathcal{M}, w \Vdash_s A \wedge B$  iff  $\mathcal{M}, w \Vdash_s A$  and  $\mathcal{M}, w \Vdash_s B$
- $\mathcal{M}, w \Vdash_t A \vee B$  iff  $\mathcal{M}, w \Vdash_t A$  or  $\mathcal{M}, w \Vdash_t B$
- $\mathcal{M}, w \Vdash_s A \vee B$  iff  $\mathcal{M}, w \Vdash_s A$  or  $\mathcal{M}, w \Vdash_s B$
- $\mathcal{M}, w \Vdash_t \neg A$  iff for all  $v$  where  $w \leq v$  we have  $\mathcal{M}, v \not\Vdash_s A$
- $\mathcal{M}, w \Vdash_s \neg A$  iff for all  $v$  where  $w \leq v$  we have  $\mathcal{M}, v \not\Vdash_t A$
- $\mathcal{M}, w \Vdash_t A \rightarrow B$  iff for all  $v$  where  $w \leq v$  we have either  $\mathcal{M}, v \not\Vdash_s A$  or  $\mathcal{M}, v \Vdash_t B$
- $\mathcal{M}, w \Vdash_s A \rightarrow B$  iff for all  $v$  where  $w \leq v$  we have either  $\mathcal{M}, v \not\Vdash_t A$  or  $\mathcal{M}, v \Vdash_s B$

It's helpful in looking at the above clauses to note that, letting  $\bar{s} = t$  and  $\bar{t} = s$ , the  $t$ - and  $s$ -satisfaction clauses for ' $\rightarrow$ ' and ' $\neg$ ' have a common form, namely:

- $\mathcal{M}, w \Vdash_x \neg A$  iff for all  $v$  where  $w \leq v$  we have  $\mathcal{M}, v \not\Vdash_{\bar{x}} A$
- $\mathcal{M}, w \Vdash_x A \rightarrow B$  iff for all  $v$  where  $w \leq v$  we have either  $\mathcal{M}, v \not\Vdash_{\bar{x}} A$  or  $\mathcal{M}, v \Vdash_x B$

We will exploit this uniformity below in our soundness & completeness theorems.

As with the Kripke semantics for intuitionistic logic it is easy to show that the above semantic clauses allow us to lift the conditions of PERSISTENCE and CONTAINMENT give above from atomic sentences to the full language.

**Lemma 1 (Persistence).** *For all formulas  $A$ , and worlds  $w, w'$  we have for  $x \in \{t, s\}$ :*

$$\text{If } \mathcal{M}, w \Vdash_x A \text{ and } w \leq w', \text{ then } \mathcal{M}, w' \Vdash_x A$$

*Proof.* By induction on the complexity of  $A$ . Basis case follows from T-PERSISTENCE or S-PERSISTENCE. The only interesting case in the induction step are those for ' $\rightarrow$ ' and ' $\neg$ '. We give the case for  $\rightarrow$  (that for  $\neg$  is similar). Suppose, then, that

<sup>5</sup> In [2] it is pointed out that the appropriate analogue of the principle of factivity in this setting is actually the classically-equivalent  $KA \rightarrow \neg\neg A$ , the addition of which to  $\mathbb{I}EL^-$  results in the logic  $\mathbb{I}EL$ .

(i)  $\mathcal{M}, w \Vdash_x A \rightarrow B$ , that (ii)  $w \leq w'$  but that (iii)  $\mathcal{M}, w' \not\Vdash_x A \rightarrow B$ . So by (iii) there is a  $w''$  where (iv)  $w' \leq w''$  and (v)  $\mathcal{M}, w'' \Vdash_{\bar{x}} A$  but (vi)  $\mathcal{M}, w'' \not\Vdash_x B$ . By (ii) and (iv) we have  $w \leq w''$ , which from (i) gives us that either  $w'' \not\Vdash_{\bar{x}} A$  or  $w'' \Vdash_x B$ . But from (v) it follows that we must have  $w'' \Vdash_x B$ , contradicting (vi). So the result follows by reductio.  $\square$

**Lemma 2 (Containment).** *For all formulas  $A$  and worlds  $w$  we have that:*

$$\mathcal{M}, w \Vdash_s A \text{ only if } \mathcal{M}, w \Vdash_t A$$

*Proof.* By induction on the complexity of  $A$ . The basis case follows from CONTAINMENT. The only interesting case in the induction step are those for  $\rightarrow$  and  $\neg$ . We give the case for  $\rightarrow$  (that for  $\neg$  is similar). Suppose that  $\mathcal{M}, w \Vdash_s A \rightarrow B$ , but  $\mathcal{M}, w \not\Vdash_t A \rightarrow B$ . Then we have a  $w'$  where  $w \leq w'$  and  $w' \Vdash_s A$  and  $w' \not\Vdash_t B$ . By the induction hypothesis we have  $w' \Vdash_t A$ . By the fact that  $\mathcal{M}, w \Vdash_s A \rightarrow B$  and  $w \leq w'$  we have that if  $w' \Vdash_t A$  then  $w' \Vdash_s B$ . So by the induction hypothesis again we have that  $w' \Vdash_t B$  giving us a contradiction.  $\square$

As mentioned above, we will be using the above model theory to characterize nonreflexive and nontransitive logics. We will do this by defining a variety of mixed notions of consequence over these models as follows.

**Definition 2.** *Suppose that  $x, y \in \{s, t\}$ . Then given a sequent  $[\Gamma \succ \Delta]$  say that it  $xy$ -holds at  $w$  in  $\mathcal{M}$  iff either  $\mathcal{M}, w \not\Vdash_x \gamma$  for some  $\gamma \in \Gamma$ , or if  $\mathcal{M}, w \Vdash_y \delta$  for some  $\delta \in \Delta$ , and that it  $xy$ -holds throughout  $\mathcal{M}$  iff it  $xy$ -holds at all worlds in  $\mathcal{M}$ . Finally we will say that a sequent is  $xy$ -valid iff it  $xy$ -holds in all models.*

This definition gives rise to the following two ‘mixed’ notions of validity.<sup>6</sup> Say that a sequent is  $\text{llts}$ -valid iff it is  $\text{ts}$ -valid, and  $\text{llst}$ -valid iff it is  $\text{st}$ -valid.

Defining validity for metasequents in this setting presents us with choices which are not present in the valuational setting adopted above and in [8] and elsewhere in the literature. In the literature on metasequent validity it is typical to draw a distinction between a *local* and *global* notion of metasequent validity where:

- A metasequent  $\Sigma \Rightarrow \sigma$  is *metasequentially locally valid* iff for all models  $m$ , if every member of  $\Sigma$  holds in  $m$ , then  $\sigma$  holds in  $m$ .
- A metasequent  $\Sigma \Rightarrow \sigma$  is *metasequentially globally valid* iff if every member of  $\Sigma$  holds in all models, then  $\sigma$  holds in all models.

<sup>6</sup> It is worth briefly noting that we also get intuitionistic analogues of the Logic of Paradox and Strong Kleene logic out of this definition as the set of all  $\text{tt}$  and  $\text{ss}$ -valid sequents respectively. These logics are not substructural, like the mixed notions, but do have a number of interesting constructive features. I investigate some curious features of the ‘Constructive Logic of Paradox’ in [9], building off work in the present paper along with previous work on the constructive analogues of Parry’s logic of analytic implication in [5], and Halldén’s logic of nonsense in [22] (on which also see [1]).

As is done in [8] we will work with local metasequential validity (although see [19] and [12] for arguments concerning whether to favor one notion over the other). In broadly modal settings like the present one there are two reasonable looking notions of local metasequential validity, determined by two different ways of regarding a sequent as holding in a model, namely the following:

- A metasequent  $\Sigma \Rightarrow \sigma$  is *local-local xy-valid* iff for all models  $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$  and every  $w \in W$ , if every member of  $\Sigma$  xy-holds at  $w$ , then  $\sigma$  xy-holds at  $w$ .
- A metasequent  $\Sigma \Rightarrow \sigma$  is *local-global xy-valid* iff for all models  $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$ , if every member of  $\Sigma$  xy-holds throughout  $\mathcal{M}$ , then  $\sigma$  xy-holds throughout  $\mathcal{M}$ .

Is there any way to make a principled choice between these two different accounts of validity in this setting? One prima facie desirable feature for an account of validity for a nonreflexive logician is that it allows one to execute the recapture strategy outlined in [7]. There it is argued that a nonreflexive logician can account for the correctness of classically valid inferences by counting them as involving suppressed metainferential premises. For example while  $[\neg\neg p \succ \neg p]$  is not TS-valid, the metainference

$$[p \succ p] \Rightarrow [\neg\neg p \succ \neg p]$$

is TS-valid, allowing the nonreflexive logician to explain the correctness of the inference  $[\neg\neg p \succ \neg p]$  by pointing out that the above metainference is logically valid, and we are often correct in suppressing the atomic metainferential premise  $[p \succ p]$  in non-paradoxical contexts. Explanations for the correctness of intuitionistically valid inferences like triple-negation elimination are unavailable to a nonreflexive logician who uses local-local validity.<sup>7</sup> As we can see, this metainference is invalid according to local-local ts-validity. Consider the following model  $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$ :

- $W = \{w, v\}$
- $\leq = \{\langle w, w \rangle, \langle w, v \rangle, \langle v, v \rangle\}$
- $V_s(p) = \emptyset, V_t(p) = \{v\}$ .

Given that  $w \not\Vdash_t p$  we have  $[p \succ p]$  ts-holds at  $w$ . But it is also clear that  $v \Vdash_t \neg p$ , as for no  $\leq$ -successor  $u$  of  $v$  do we have  $u \Vdash_s p$ , and so all  $\leq$ -successors  $u$  of  $w$  are such that  $u \not\Vdash_s \neg\neg p$ , from which we have  $w \Vdash_t \neg\neg\neg p$ . At the same time  $w \not\Vdash_s \neg p$  as we have  $v \Vdash_t p$ . So  $[\neg\neg\neg p \succ \neg p]$  fails to ts-hold at  $w$ . So this metainference has a local-local counterexample at  $w$ . By contrast, it is easy to see that the above metainference is local-global valid—the above counterexample being ruled out by the fact that  $[p \succ p]$  does not ts-hold at  $v$ , this feature being crucial to the local-local counterexample to this metainference.

<sup>7</sup> Thank you to Pablo Cobreros for correspondence and conversations which helped me more clearly think through this point.

Given this we will in the rest of this paper adopt local-global metainferential validity, and whenever we talk of a metasequent being  $xy$ -valid what we mean is that the metasequent in question is local-global  $xy$ -valid.

One immediate use of the above machinery which we can make at this point is to show that  $\mathbb{L}ts$  and  $\mathbb{L}st$  are respectively  $[Id]$ - and  $[Cut]$ -companions of  $\mathbb{L}$ . We will treat the case of  $\mathbb{L}ts$  in detail below, the case of  $\mathbb{L}st$  following similarly. First we will want to make use of the following Lemma.

**Lemma 3.** *A model  $\langle W, \leq, V_s, V_t \rangle$   $ts$ -validates all instances of  $[Id]$  iff  $V_s = V_t$ .*

*Proof.* For the ‘only if’ direction we prove the contrapositive. Suppose that  $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$  and  $V_s \neq V_t$ , so for some  $p_i$  we have  $w \in V_t(p_i)$  but  $w \notin V_s(p_i)$  (as by CONTAINMENT if we have  $w \in V_s(p_i)$  we also have  $w \in V_t(p_i)$ ). From this it follows that  $w \Vdash_t p_i$  but  $w \not\Vdash_s p_i$  and so it follows directly that we do not have  $\mathcal{M} \models_{ts} [p_i \succ p_i]$ , and so that instance of  $[Id]$  is not validated in  $\mathcal{M}$ .

For the ‘if’ direction suppose  $V_s = V_t$ . We prove by induction on the complexity of  $A$  that all instances of  $[A \succ A]$  hold in  $\mathcal{M}$ . For the basis case suppose that  $w \Vdash_t p_i$ , and so  $w \in V_t(p_i)$ . As, by hypothesis,  $V_t(p_i) = V_s(p_i)$  it follows that  $w \in V_s(p_i)$  and so  $w \Vdash_s p_i$ . But  $w$  was arbitrary so at every world  $w \in W$  if  $w \Vdash_t p_i$  then  $w \Vdash_s p_i$ , and so  $\mathcal{M} \models_{ts} [p_i \succ p_i]$ . For the induction hypothesis suppose, then, that for all formulas  $B$  of complexity less than  $n$  we have that  $\models_{ts} [B \succ B]$ . Suppose, then, that  $w \Vdash_t \neg B$ . Then for all  $w \leq v$  we have  $v \not\Vdash_s B$ . But we have  $\models_{ts} [B \succ B]$ , so this means that at all such  $v$  we must have  $v \not\Vdash_t B$ , and so it follows that we have  $w \Vdash_s \neg B$ , as desired.  $\square$

**Theorem 2.** *Intuitionistic Logic is the  $[Id]$ -companion of  $\mathbb{L}ts$ , and the  $[Cut]$ -companion of  $\mathbb{L}st$ .*

*Proof.* As stated above we’ll treat only the  $[Id]$ -case here. Recall that what we need to demonstrate to show that Intuitionistic logic is the  $[Id]$ -companion of  $\mathbb{L}ts$  is (a) that  $[Id]$  is not valid in  $\mathbb{L}ts$ , and (b) that the addition of all instances of  $[Id]$  to  $\mathbb{L}ts$  results in intuitionistic logic. It is easy to show that not all instances of  $[Id]$  are valid in  $\mathbb{L}ts$ . To see this consider the following simple model  $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$ .

- $W = \{w\}$
- $\leq = \{\langle w, w \rangle\}$
- $V_s(p_i) = \emptyset$
- $V_t(p_i) = W$

In this model we have  $w \Vdash_t p_i$ , but  $w \not\Vdash_s p_i$  for all  $w \in W$ , from which it follows directly that  $[p_i \succ p_i]$  does not  $ts$ -hold throughout  $\mathcal{M}$ , and so the metainference  $\Rightarrow [p_i \succ p_i]$  is not  $ts$ -valid.

Now we show that restricting ourself to models which validate all instances of  $[Id]$  results in intuitionistic logic. From Lemma 3 we have that the class of all models which validate all instances of  $[Id]$  are those models of the form  $\langle W, \leq, V, V \rangle$ , each of which are trivially isomorphic to Kripke models for intuitionistic logic, which from [13] we know determine intuitionistic logic.  $\square$

### 3 Proof-Theory

#### 3.1 A Metasequent Calculus

Here we give a proof system which we label **SubIL**, which is a general calculus for determining the validity of metasequents over the models defined in the previous section. This system is a metasequent calculus in the style of [8], built of Maehara's multiple-conclusion sequent calculus for intuitionistic logic, but with the addition of labels in the style of [4]. All formulas in our sequents will be labelled with truth-statuses—being of the form either  $t : A$  or  $s : A$ . In the following rules, recall that  $x \in \{s, t\}$  and  $\bar{s} = t$  and  $\bar{t} = s$ . Given a sequent  $[\Gamma \succ \Delta]$  let  $[\Gamma \succ \Delta]^{\text{xy}}$  be the sequent  $[\Gamma^{\text{x}} \succ \Delta^{\text{y}}]$  where  $\Gamma^{\text{x}} = \{x : A \mid A \in \Gamma\}$  and  $\Delta^{\text{y}} = \{y : A \mid A \in \Delta\}$ . Due to the addition of labels this metasequent calculus gives us a general calculus for demonstrating that a metainference  $\sigma_1, \dots, \sigma_n \Rightarrow \sigma$  is  $\text{xy}$ -valid by producing a derivation of the labelled metasequent  $\sigma_1^{\text{xy}}, \dots, \sigma_n^{\text{xy}} \Rightarrow \sigma^{\text{xy}}$ .

Our calculus will consist of four different families of rules: outer structural rules (governing the structure of metasequents), inner structural rules (governing the structure of sequents), premise operational rules (governing the introduction of connectives in sequents in our metasequent premises), and conclusion operational rules (governing the introduction of connectives our metasequents conclusion sequent). We describe each of these rules in order below.

##### OUTER STRUCTURAL RULES

$$\frac{}{\Sigma, \sigma \Rightarrow \sigma} \text{[ID]} \quad \frac{\Sigma \Rightarrow \sigma' \quad \Sigma', \sigma' \Rightarrow \sigma}{\Sigma, \Sigma' \Rightarrow \sigma} \text{[CUT]}$$

##### INNER STRUCTURAL RULES

$$\frac{}{\Sigma \Rightarrow [s : A \succ t : A]} \text{[id}_{st}]} \quad \frac{}{\Sigma \Rightarrow [s : A \succ s : A]} \text{[id}_{ss}]}$$

$$\frac{}{\Sigma \Rightarrow [t : A \succ t : A]} \text{[id}_{tt}]}$$

$$\frac{\Sigma \Rightarrow [\Gamma \succ \Delta, x : A] \quad \Sigma \Rightarrow [x : A, \Gamma' \succ \Delta']}{\Sigma \Rightarrow [\Gamma, \Gamma' \succ \Delta, \Delta']} \text{[cut]} \quad \frac{\Sigma \Rightarrow [\Gamma \succ \Delta]}{\Sigma \Rightarrow [\Gamma, \Gamma' \succ \Delta, \Delta']} \text{[dil]}$$

##### CONCLUSION OPERATIONAL RULES

$$\frac{\Sigma \Rightarrow [\Gamma \succ \Delta, x : A] \quad \Sigma \Rightarrow [\Gamma \succ \Delta, x : B]}{\Sigma \Rightarrow [\Gamma \succ \Delta, x : A \wedge B]} \text{[}\wedge R_c^x\text{]} \quad \frac{\Sigma \Rightarrow [\Gamma, x : A, x : B \succ \Delta]}{\Sigma \Rightarrow [\Gamma, x : A \wedge B \succ \Delta]} \text{[}\wedge L_c^x\text{]}$$

$$\frac{\Sigma \Rightarrow [\Gamma \succ \Delta, x : A, x : B]}{\Sigma \Rightarrow [\Gamma \succ \Delta, x : A \vee B]} \text{[}\vee R_c^x\text{]} \quad \frac{\Sigma \Rightarrow [\Gamma, x : A \succ \Delta] \quad \Sigma \Rightarrow [\Gamma, x : B \succ \Delta]}{\Sigma \Rightarrow [\Gamma, x : A \vee B \succ \Delta]} \text{[}\vee L_c^x\text{]}$$

$$\frac{\Sigma \Rightarrow [\Gamma, \bar{x} : A \succ x : B]}{\Sigma \Rightarrow [\Gamma \succ x : A \rightarrow B]} \text{[}\rightarrow R_c^x\text{]} \quad \frac{\Sigma \Rightarrow [\Gamma \succ \Delta, \bar{x} : A] \quad \Sigma \Rightarrow [x : B, \Gamma \succ \Delta]}{\Sigma \Rightarrow [\Gamma, x : A \rightarrow B \succ \Delta]} \text{[}\rightarrow L_c^x\text{]}$$

$$\frac{\Sigma \Rightarrow [\Gamma, \bar{x} : A \succ]}{\Sigma \Rightarrow [\Gamma \succ x : \neg A]} \text{[}\neg R_c^x\text{]} \quad \frac{\Sigma \Rightarrow [\Gamma \succ \Delta, \bar{x} : A]}{\Sigma \Rightarrow [\Gamma, x : \neg A \succ \Delta]} \text{[}\neg L_c^x\text{]}$$

## PREMISE OPERATIONAL RULES

$$\begin{array}{c}
\frac{\Sigma, [\Gamma \succ \Delta, x : A], [\Gamma \succ \Delta, x : B] \Rightarrow \sigma}{\Sigma, [\Gamma \succ \Delta, x : A \wedge B] \Rightarrow \sigma} \quad [\wedge R_p^x] \qquad \frac{\Sigma, [\Gamma, x : A, x : B \succ \Delta] \Rightarrow \sigma}{\Sigma, [\Gamma, x : A \wedge B \succ \Delta] \Rightarrow \sigma} \quad [\wedge L_p^x] \\
\\
\frac{\Sigma, [\Gamma \succ \Delta, x : A, x : B] \Rightarrow \sigma}{\Sigma, [\Gamma \succ \Delta, x : A \vee B] \Rightarrow \sigma} \quad [\vee R_p^x] \qquad \frac{\Sigma, [\Gamma, x : A \succ \Delta], [\Gamma, x : B \succ \Delta] \Rightarrow \sigma}{\Sigma, [\Gamma, x : A \vee B \succ \Delta] \Rightarrow \sigma} \quad [\vee L_p^x] \\
\\
\frac{\Sigma, [\Gamma, \bar{x} : A \succ x : B] \Rightarrow \sigma}{\Sigma, [\Gamma \succ x : A \rightarrow B] \Rightarrow \sigma} \quad [\rightarrow R_p^x] \qquad \frac{\Sigma, [\Gamma, x : A \rightarrow B \succ \Delta, \bar{x} : A], [x : B, \Gamma \succ \Delta] \Rightarrow \sigma}{\Sigma, [\Gamma, x : A \rightarrow B \succ \Delta] \Rightarrow \sigma} \quad [\rightarrow L_p^x] \\
\\
\frac{\Sigma, [\Gamma, \bar{x} : A \succ] \Rightarrow \sigma}{\Sigma, [\Gamma \succ x : \neg A] \Rightarrow \sigma} \quad [\neg R_p^x] \qquad \frac{\Sigma, [\Gamma, x : \neg A \succ \Delta, \bar{x} : A] \Rightarrow \sigma}{\Sigma, [\Gamma, x : \neg A \succ \Delta] \Rightarrow \sigma} \quad [\neg L_p^x]
\end{array}$$

## 4 Soundness &amp; Completeness

We semantically interpret our labelled calculus in the expected way, adopting the notation that  $\mathcal{M} \models_w x : A$  is to mean  $\mathcal{M}, w \Vdash_x A$ . The definitions of what it is for a labelled sequent to hold in a model, and for a metasequent to hold in a model are the obvious elaborations of the definitions given above. A labelled sequent  $[\Gamma \succ \Delta]$  holds in at  $w$  in a model  $\mathcal{M}$  iff either  $\mathcal{M} \not\models_w \gamma$  for some  $\gamma \in \Gamma$  or  $\mathcal{M} \models_w \delta$  for some  $\delta \in \Delta$ . A meta-inference  $\Sigma \Rightarrow \sigma$  of labelled sequents is *valid* iff for all models  $\mathcal{M}$ , if every  $\sigma' \in \Sigma$  holds at all worlds  $w \in W$ , then  $\sigma$  holds at all worlds  $w \in W$ .

**Theorem 3 (Soundness).** *If a metasequent  $\Sigma \Rightarrow \sigma$  is derivable in SubL, then  $\Sigma \Rightarrow \sigma$  is valid.*

*Proof.* We proceed by induction on the structure of derivations, showing that our initial sequents are valid and our rules preserve validity. For the basis case it is easy to see that  $[ID]$ ,  $[id_{ss}]$  and  $[id_{tt}]$  hold in all models, and that  $[id_{st}]$  holds in all models follows from Lemma 2.

We give an illustrative selection of cases for the induction step below.

**[cut]** Suppose that the last rule applied is  $[cut]$ , and so our derivation ends like this:

$$\frac{\Sigma \Rightarrow [\Gamma \succ \Delta, x : A] \quad \Sigma \Rightarrow [x : A, \Gamma' \succ \Delta']}{\Sigma \Rightarrow [\Gamma, \Gamma' \succ \Delta, \Delta']} \quad [cut]$$

Suppose that the conclusion sequent fails to hold at some world  $w \in W$ , and so we have  $\models_w \Gamma \cup \Gamma'$ , but  $\not\models_w \delta$  for any  $\delta \in \Delta \cup \Delta'$ . Now either (i)  $w \Vdash_x A$  or (ii)  $w \not\Vdash_x A$ . If (i) then the RHS premise fails to hold. If (ii) then the LHS premise fails to hold. So either way one of the premises will fail to hold if the conclusion fails to do so.

$[\rightarrow \mathbf{R}_c^x]$ : Suppose that the last rule applied is  $[\rightarrow R_c^x]$ , and so our derivation ends like this:

$$\frac{\Sigma \Rightarrow [\Gamma, \bar{x} : A \succ x : B]}{\Sigma \Rightarrow [\Gamma \succ x : A \rightarrow B]} \quad [\rightarrow R_c^x]$$

Suppose that the conclusion sequent fails to hold in  $\mathcal{M}$ . Then there is some  $w \in W$  s.t.  $\mathcal{M} \models_w \Gamma$  but  $w \not\models_x A \rightarrow B$ . So there  $\exists w \leq w'$  s.t.  $w' \Vdash_{\bar{x}} A$  and  $w' \not\models_x B$ . By Lemma 1 we also have  $\mathcal{M} \models_{w'} \Gamma$  and so the premise sequent fails to hold at  $w'$ .

$[\rightarrow \mathbf{L}_p^x]$ : Suppose that the last rule applied is  $[\rightarrow L_p^x]$ , and so our derivation ends like this:

$$\frac{\Sigma, [\Gamma, x : A \rightarrow B \succ \Delta, \bar{x} : A], [x : B, \Gamma \succ \Delta] \Rightarrow \sigma}{\Sigma, [\Gamma, x : A \rightarrow B \succ \Delta] \Rightarrow \sigma} \quad [\rightarrow L_p^x]$$

Suppose that the conclusion sequent fails to hold. In particular this means that for all  $w \in W$  we have every metasequent premise holding at  $w$  (while  $\sigma$  fails at some  $w$ ). So suppose that we have  $\mathcal{M} \models [\Gamma, x : A \rightarrow B \succ \Delta]$ , and suppose that we have  $\mathcal{M} \models_w \Gamma$ ,  $\mathcal{M} \not\models_w \delta$  for all  $\delta \in \Delta$ . Then we must have  $\mathcal{M} \not\models_w x : A \rightarrow B$  (automatically making the LHS premise sequent hold at  $w$ ). So there is some  $w \leq w'$  such that  $w' \Vdash_{\bar{x}} A$  and  $w' \not\models_x B$ . So by persistence we have  $w \not\models_x B$ , and so the RHS premise holds at  $w$ .  $\square$

**Theorem 4 (Completeness).** *If  $\Sigma \Rightarrow \sigma$  is underivable in SubL, then it is invalid.*

*Proof.* The countermodel construction we present here is a combination of that used in [8] and that for intuitionistic logic used in [18]. We proceed in stages as outlined below.

*Constructing Reduction Trees* Suppose that we have an enumeration  $x : A_1, x : A_2, \dots, x : A_n, \dots$  of all labelled formulas. Let  $\Sigma \Rightarrow \sigma$  be a metasequent. We will construct a reduction tree of metasequents as follows.

- The root of our reduction tree is the metasequent  $\Sigma \Rightarrow \sigma$
- A branch of our tree is *closed* iff (i) its leaf metasequents are of the form  $\Theta, [X \succ Y] \Rightarrow [X, X' \succ Y, Y']$  (in which case we can derive this metasequent by applying  $[dil]$  to an instance of  $[ID]$ ), (ii) its leaf metasequents are of them form  $\Sigma \Rightarrow [X, s : A \succ Y, t : A]$ ,  $\Sigma \Rightarrow [X, s : A \succ Y, s : A]$ , or  $\Sigma \Rightarrow [X, t : A \succ Y, t : A]$  (in which case we can derive this metasequent by applying  $[dil]$  to an instance of  $[id_{st}]$ ,  $[id_{ss}]$ , or  $[id_{tt}]$  respectively). A branch which is not closed is *open*.
- In the first stage we process our metasequent premises. To do this we extend our open branch by applying root-first one our of PREMISE OPERATIONAL RULES to a sequent  $\theta_i \in \Theta$ , where  $\Theta \Rightarrow \theta$  is an open leaf metasequent of our tree. So, for example, if our leaf metasequent is of the form

$\Sigma, [\Gamma, x : A \rightarrow B \succ \Delta] \Rightarrow \sigma$  and we apply  $[\rightarrow L_p^x]$  root-first expanding out tree to give us a new leaf metasequent of

$$\Sigma, [\Gamma, x : A \rightarrow B \succ \Delta, \bar{x} : A], [\Gamma, x : B \succ \Delta] \Rightarrow \sigma$$

- We continue doing this until this process reaches a fixed point.

At the end of this stage we will have ended up with a tree of metasequents the root of which is  $\Sigma \Rightarrow \sigma$ . Let  $\Sigma^+$  be the result of taking the union of all the  $\Sigma_i$  such that  $\Sigma_i \Rightarrow \sigma$  appears as a node in the tree constructed in our first stage. We will then create our reduction tree using the metasequent  $\Sigma^+ \Rightarrow \sigma$ .

- To create a reduction tree we proceed as follows. Suppose that  $\Theta \Rightarrow [X \succ Y]$  is a leaf metasequent of an open branch. We then apply the  $[\wedge L_c^x]$ ,  $[\wedge R_c^x]$ ,  $[\vee L_c^x]$ ,  $[\vee R_c^x]$ ,  $[\rightarrow L_c^x]$ ,  $[\rightarrow R_c^x]$  rules root-first to our leaf metasequent, guided by the order in which formulas appear in our enumeration. We perform these reductions until there are no longer any formulas in the leaf metasequents of open branches to which we are able to apply our reduction which we have not yet applied it to.
- Once there are no longer any formulas in the leaf of an open branch which we can apply our reductions to we expand that branch with two new leaf nodes  $\Theta \Rightarrow [X \succ Y, x : A_i]$  and  $\Theta \Rightarrow [x : A_i, X \succ Y]$  where  $x : A_i$  is the next formula in our enumeration. This is our *[cut]*-reduction.
- An open branch is *completed* iff there is no longer any reduction which can be applied to it.

Let us refer to the completed reduction tree for a metasequent  $S$  as  $T(S)$ . It is a routine matter to see that if all of the leaf metasequents of  $T(S)$  are provable in **SubIL** then so is  $S$ , given the manner in which the construction of our reduction tree is guided by the rules of **SubIL**. In particular, then, if  $S$  is unprovable then there must be some branch, call it  $B_S$ , in  $T(S)$  every metasequent of which is unprovable. We will assign the metasequent which results from merging together all the metasequents on  $B_{S_0}$  (where  $S_0 = \Sigma^+ \Rightarrow \sigma$ ) the label 0.

We now move on to the third stage of our construction. Given a metasequent  $\Theta \Rightarrow [X_n \succ Y_n]$  labelled  $n$  which results from merging together all the sequents on  $B_{S_n}$ , consider for each formula  $x : B \rightarrow C \in Y_n$  (and similarly for  $x : \neg B \in Y_n$ ) the metasequents:

$$\Theta \Rightarrow [X, \bar{x} : B \succ x : C]$$

It is easy to see that this sequent cannot be **SubIL** provable (otherwise we could prove  $\Theta \Rightarrow [X \succ Y]$  by applying  $[\rightarrow R_c^x]$  and then *[dil]*). Let us label this new sequent  $nm$ , where  $m$  is the position of  $x : B \rightarrow C$  in our enumeration, and let  $T(S_{nm})$  be its reduction tree. Continue this procedure by selecting an open branch  $B_{S_{nm}}$  and for each formula of the form  $x : C \rightarrow D \in Y_{nm}$  (similarly  $x : \neg C \in Y_{nm}$ ) considering the metasequent  $\Theta \Rightarrow [X_{nm}, \bar{x} : C \succ x : D]$ , calling this sequent  $nml$  where  $l$  is the position of  $x : C \rightarrow D$  in our enumeration.

Continue this procedure  $\omega$  many times.

*Converting the Reduction Trees into a Model:* We will use the collection of all of these reduction trees to define our model as follows, referring to the metasequent with label  $l$  as  $\Theta \Rightarrow [X_l \succ Y_l]$ . Let  $W$  be the set of all labels given to metasequents, and for  $l, m \in W$  set  $l \leq m$  iff  $l$  is an initial segment of  $m$ . Define the  $V_p$  and  $V_s$  as follows:

- $w \in V_t(p_i)$  iff  $t : p \in X_w$ .
- $w \in V_s(p_i)$  iff  $s : p \in X_w$ .

It is easy to see by construction that the model defined here satisfies the conditions of T-PERSISTENCE and S-PERSISTENCE. To see that it satisfies CONTAINMENT suppose that (i)  $w \in V_s(p_i)$  and suppose (ii)  $w \notin V_t(p_i)$ . By (i) we know that  $s : p_i \in X_w$ , and by (ii) that  $t : p_i \notin X_w$ . As  $\Theta \Rightarrow [X_w \succ Y_w]$  is the merge of an open branch it follows from the *[cut]*-reduction that we must have  $t : p_i \in Y_w$ . But this would mean that sequent  $w$  is of the form  $\Theta \Rightarrow [X_w, s : p_i \succ Y_w, t : p_i]$ , and hence is closed! So it follows that CONTAINMENT must be satisfied.

*Showing that truth and membership coincide:* Having shown that this is indeed a model, we now move on to showing that  $x : A \in X_w$  iff  $w \Vdash_x A$ . The basis case is handled by our definition of  $V_s$  and  $V_t$ . Suppose, then, that for all formulas  $B$  with complexity  $< n$  we have  $x : B \in X_w$  iff  $w \Vdash_x B$ . We deal with the two directions for  $\rightarrow$  in the inductions step, the others being similar:

- Suppose that  $x : B \rightarrow C \in X_w$ . Then by construction for all  $v$  such that  $w \leq v$  we have  $x : B \rightarrow C \in X_v$ . So by the  $[\rightarrow L]$ -reduction we have either  $\bar{x} : B \in Y_v$  or  $x : C \in X_v$ . So by the IH we have either  $v \not\Vdash_{\bar{x}} B$  or  $v \Vdash_x C$ . As this holds for all  $w \leq v$  it follows, then, that  $w \Vdash_x B \rightarrow C$ .
- Suppose that  $x : B \rightarrow C \notin X_w$ . By the *[cut]*-reduction we have that  $x : B \rightarrow C \in Y_w$ . So by the second stage of construction there is some  $v$  such that  $w \leq v$  where  $\bar{x} : B \in X_v$  and  $x : C \in Y_v$ . So by the IH we have  $v \Vdash_{\bar{x}} B$  and  $v \not\Vdash_x C$ . So it follows that  $w \not\Vdash_x B \rightarrow C$ .

*Showing that our model is a counterexample:* Now we need to show that this model is such that, at our root node 0 we have  $\Sigma \Rightarrow [\Gamma \succ \Delta]$  failing to hold. It is easy to see that at 0 as  $\Gamma \subseteq X_0$  we have  $\mathcal{M} \models_0 \Gamma$ . Suppose, then, that for some  $x : A \in \Delta$  we have  $\mathcal{M} \models_0 x : A$ . Then we must have  $x : A \in X_0$ , but as  $\Delta \subseteq Y_0$  we must also have  $x : A \in Y_0$ , meaning that our branch is closed! So for all  $x : A \in \Delta$  we have  $\mathcal{M} \not\models_0 x : A$ . Suppose, then, that  $[X \succ Y] \in \Sigma$ . As none of the branches  $B_S$  are closed we must have, for each metasequent  $\Theta \Rightarrow [X_w \succ Y_w]$  with label  $w$ , that either  $X \not\subseteq X_w$  or  $Y \not\subseteq Y_w$ , and so either some  $x : A \in X$  where  $x : A \notin X_w$ , or some  $x : B \in Y$  where  $x : B \notin Y_w$ . For the first if  $x : A \notin X_w$  we must have  $w \not\Vdash_x A$  and so the sequent holds at  $w$ . For the second if  $B \notin Y_w$  then we must have  $B \in X_w$  by *[cut]*-closure, and so  $w \Vdash_x B$ , and so the sequent holds at  $w$ . As this is the case for all  $w$  in the model, it follows that all the metasequent-premises hold at all worlds, while  $[\Gamma \succ \Delta]$  fails at 0, and thus the sequent is invalid.  $\square$

Our completeness result brings with it some corollaries related to our different notions of metasequent validity. For example, it follows from the above results that a metasequent  $\Sigma \Rightarrow \sigma$  is **ts**-valid iff  $\Sigma^{\text{ts}} \Rightarrow \sigma^{\text{ts}}$  is derivable in **SubIL**. We can summarize these results as follows.

**Corollary 1.** *A metasequent  $\Sigma \Rightarrow \sigma$  is **xy**-valid iff  $\Sigma^{\text{xy}} \Rightarrow \sigma^{\text{xy}}$  is **SubIL** derivable.*

## 5 Relating Classical and Constructive Substructural Logics

We close by looking at the relationship between **ILts** and **ILst** and **TS** and **ST**. It is well known, and easy to show, that classical logic is determined by the class of all intuitionistic Kripke frames whose accessibility relation is the identity relation, and that the law of excluded middle— $A \vee \neg A$ —corresponds (as shown in [21, p.234]) to this class of frames.<sup>8</sup>

It is easy to show that a similar determination fact obtains in the case of our substructural Kripke frames. The most straightforward way to see this is through the following connection between our semantics in terms of substructural Kripke models, and the semantics for **TS** and **ST** given in terms of trivaluations in §1.

For convenience let  $\mathcal{ID}$  be the class of all substructural Kripke models where  $\leq$  is just the identity relation on  $W$ .

**Proposition 1.** *1. Let  $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$  be a model in  $\mathcal{ID}$ , and  $w \in W$ , and define the trivaluation  $v_w$  as follows:*

$$v_w(A) = \begin{cases} 1 & w \Vdash_s A \\ i & w \Vdash_t A \text{ and } w \not\Vdash_s A \\ 0 & w \not\Vdash_t A \end{cases}$$

*Then the trivaluation  $v_w$  is a Strong Kleene valuation.*

*2. Let  $v$  be a Strong Kleene valuation. Then  $\langle \{w\}, \{\langle w, w \rangle\}, V_s^v, V_t^v \rangle$  is a model, where  $w \in V_s^v(p_i)$  iff  $v(p_i) = 1$ , and  $w \in V_t^v(p_i)$  iff  $v(p_i) \in \{1, i\}$ .*

**Corollary 2.** *A metasequent  $\Sigma \Rightarrow \sigma$  is **XY**-valid iff it is global-local **xy**-valid in the class of all models in  $\mathcal{ID}$ .*

The question then arises: what metasequents correspond to the class of  $\mathcal{ID}$  models? Interestingly this is sensitive to what species of **xy**-validity we are using. First say that a metasequent  $\Sigma \Rightarrow \sigma$  is *xy-valid on a frame*  $\langle W, \leq \rangle$  iff for any model  $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$  if every sequent in  $\Sigma$  **xy**-holds throughout  $\mathcal{M}$ , then  $\sigma$  holds throughout  $\mathcal{M}$ . This is just the standard notion of validity on a frame lifted to the present setting.

<sup>8</sup> See section 3 of Segerberg's tour-de-force [17] for further discussion of determination results over minimal logic for related classes of frames.

**Theorem 5.** 1. Every instance of the metasequent  $MEM$  (*‘Metasequent Excluded Middle’*)

$$(MEM) \quad [\Gamma, A \succ \Delta], [\Gamma, \neg A \succ \Delta] \Rightarrow [\Gamma \succ \Delta]$$

is **ts**-valid on a frame  $\langle W, \leq \rangle$  iff  $\leq$  is the identity relation on  $W$ .

2. Every instance of the metasequent  $SEM$  (*‘Sequent Excluded Middle’*)

$$(SEM) \quad \Sigma \Rightarrow [\Gamma \succ \Delta, A, \neg A]$$

is **st**-valid on a frame  $\langle W, \leq \rangle$  iff  $\leq$  is the identity relation on  $W$ .

*Proof.* We treat only case 1, case 2 being very similar to the standard case for intuitionistic logic.

For the ‘only if’ direction we prove the contrapositive. Suppose that we have a frame  $\langle W, \leq \rangle$  where  $w \leq w'$  and  $w \neq w'$ . We will show that the following instance of  $MEM$  is **ts**-invalid on that frame.

$$[p \succ q], [\neg p \succ q] \Rightarrow [\succ q]$$

To do this consider the model  $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$  on the frame  $\langle W, \leq \rangle$  where  $V_t(p) = V_s(p) = V_t(q) = V_s(q) = W \setminus \{u : u \leq w\}$ . In this model we have  $u \Vdash_{ts} [p \succ q]$  for all worlds  $u \in W$ , as at all worlds  $u$  where  $u \not\leq w$ , we have  $u \Vdash_s q$ , while at worlds  $u$  where  $u \leq w$  we have  $u \not\Vdash_t p$ . We also have  $u \Vdash_{ts} [\neg p \succ q]$ , as at all worlds  $u$  we have  $u \not\Vdash_t \neg p$ , as for all those worlds  $u \leq w'$  we have  $w' \Vdash_s p$  and so  $u \not\Vdash_t \neg p$ , while for all worlds  $u \not\leq w'$  we have  $u \leq u$  and  $u \Vdash_s p$  and so  $u \not\Vdash_t \neg p$ . So both premise sequents **ts**-hold at all worlds in  $W$ . But it is clear that  $w \not\Vdash_{ts} [\succ q]$  as  $w \notin V_s(q)$ . So if  $\leq$  is not the identity relation on  $W$ , then  $MEM$  is not **ts**-valid on that frame.

For the ‘if’ direction suppose that we have a model  $\mathcal{M} = \langle W, \leq, V_s, V_t \rangle$  on a frame  $\langle W, \leq \rangle$  where  $\leq$  is the identity relation on  $W$ . Suppose further that at some world  $w \in W$  we have  $w \not\Vdash_{ts} [\Gamma \succ \Delta]$ . So we have (i)  $w \Vdash_t \gamma$  for all  $\gamma \in \Gamma$ , (ii)  $w \not\Vdash_s \delta$  for all  $\delta \in \Delta$ . Suppose then that  $w \Vdash_t A$ . Then by (i) and (ii) it follows that  $w \not\Vdash_{ts} [\Gamma, A \succ \Delta]$ . Suppose, then, that  $w \not\Vdash_t A$ . By containment it follows then that  $w \not\Vdash_s A$ , and so as  $\leq$  is the identity relation it follows that  $w \Vdash_t \neg A$ . So, then, by (i) and (ii) we have  $w \not\Vdash_{ts} [\Gamma, \neg A \succ \Delta]$ . So either way we have one of  $[\Gamma, A \succ \Delta]$  or  $[\Gamma, \neg A \succ \Delta]$  failing to hold at  $w$  when  $[\Gamma \succ \Delta]$  fails to hold at  $w$ , and so by contraposition it follows that at all worlds in this model that if  $[\Gamma, A \succ \Delta]$  and  $[\Gamma, \neg A \succ \Delta]$  hold at that world so does  $[\Gamma \succ \Delta]$ . As this model was arbitrary it then follows that  $MEM$  is valid on this frame, as desired.  $\square$

What is interesting here is that the metasequent  $[\Gamma, A \succ \Delta], [\Gamma, \neg A \succ \Delta] \Rightarrow [\Gamma \succ \Delta]$  has an **st**-counterexample in the class of  $\mathcal{ID}$  models, and similarly that  $\Sigma \Rightarrow [\Gamma \succ \Delta, A, \neg A]$  has a **ts**-counterexample in that class.<sup>9</sup> This is a further

<sup>9</sup> For the **st**-counterexample consider the model  $\langle \{w\}, =, V_s, V_t \rangle$  where  $V_s(p) = \emptyset$  and  $V_t(p) = W$ ,  $V_s(q) = V_t(q) = W$ ,  $V_s(r) = V_t(r) = \emptyset$ —then in this model the instance  $[q, p \succ r], [q, \neg p \succ r] \Rightarrow [q \succ r]$  has a **st**-counterexample. For the **ts**-counterexample consider the model  $\langle \{w\}, =, V_s, V_t \rangle$  where  $V_s(p) = \emptyset$  and  $V_t(p) = W$ —then in this model the instance  $\Rightarrow [p \succ \neg p]$  has a **ts**-counterexample.

chapter in a well-known cautionary tale where classically equivalent principles can come apart quite dramatically in non-classical logics. In this case we can see that in the context of nontransitive and nonreflexive logics whether a principle is expressed as a *rule* (or multi-premise metainference) or an *axiom* (or zero premise metainference) can make a significant difference, with the rule-version of excluded middle being equivalent in **ST** to  $[Cut]$  and the axiom-version being equivalent in **TS** to  $[Id]$ .

## 6 Conclusion

As stated in our title, this has been a mere invitation to constructive nonreflexive and nontransitive logics. Many puzzles remain, and there is much work still to be done. To close out this paper let me mention a few of the more salient and interesting puzzles which remain.

- **PROOF-THEORY**: Our brief proof-theoretic investigation of these logics required us to use a labelled metasequent proof-theory. A number of questions remain open here (aside from the obvious proof-theoretic questions about Cut elimination etc. which are also still unaddressed for the metasequent calculi for **TS** and **ST** from [8]). The obvious one is whether unlabelled metasequent proof-theory can be given for **ILts** and **ILst** (or perhaps extensions thereof).
- **SEMANTICS**: In one of the few papers which I know of that deals with intuitionistic versions of substructural logics, [20], we are presented with a very different semantic clause for the conditional (with negation being defined in terms of the conditional). Put in the notation of the present paper their clause is:
  - If  $\mathcal{M}, w \Vdash_s A \rightarrow B$  and  $\mathcal{M}, w \Vdash_t A$ , then  $\mathcal{M}, w \Vdash_s B$ .
  - $\mathcal{M}, w \Vdash_t A \rightarrow B$  iff for all  $w \leq v$ , either  $\mathcal{M}, v \not\Vdash_s A$ , or  $\mathcal{M}, v \Vdash_t B$

Note that this is a *partial* clause. Using this clause they are able to prove that their semantics is sound and complete w.r.t. the derivable sequents in a minor modification of the SET-FMLA sequent calculus **G1i**. We leave a more systematic investigation of the metainferences that are validated by this semantics, and a comparison with the metainferences validated by **ILst** for future work.

- **INTERPRETATION**: We briefly discussed above the similarities between the current semantic framework and the framework of intuitionistic epistemic logic of [2]. What metainferences are validated in the nonreflexive logic we get if we interpret valid sequents as taking us from intuitionistically true premises to intuitionistically known conclusions, or the nontransitive logic we get if we interpret valid sequents as taking us from intuitionistically known premises to intuitionistically true conclusions?

We leave the further consideration of these, and other, questions for future work.

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